

TRIMMING THE PERMUTAHEDRON TO EXTEND THE PARKING SPACE

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ABSTRACT. Berget and Rhoades asked whether the permutation representation obtained by the action of S_{n-1} on parking functions of length $n-1$ can be extended to a permutation action of S_n . We answer this question in the affirmative. We realize our module in two different ways. The first description involves binary Lyndon words and the second involves the action of the symmetric group on the lattice points of the trimmed standard permutahedron.

1. INTRODUCTION

Recall that a parking function of length $n-1$ is a sequence of nonnegative integers (a_1, \dots, a_{n-1}) whose nondecreasing rearrangement (b_1, \dots, b_{n-1}) satisfies $b_i \leq i-1$. It is immediate that the set of parking functions of length $n-1$ carries a permutation action of S_{n-1} . This parking function representation, henceforth referred to as Park_{n-1} , was first studied by Haiman in his influential work [Hai94]. In their study of an extension of Park_{n-1} , Berget and Rhoades asked [BR14, Section 4] whether the permutation action of S_{n-1} on parking functions of length $n-1$ could be extended to a permutation action of S_n . It is worth remarking that the study of ‘hidden’ actions of a larger symmetric group on spaces where a smaller symmetric group acts naturally has received a fair amount of attention in recent years; see for instance [ER19, Gai96, dG10, Mat96, Whi97]. It is in this broader context that our work is set.

The extension V_{n-1} in [BR14] is realized by considering the \mathbb{C} -span of a distinguished set of polynomials in n variables first studied by Postnikov and Shapiro [PS04]. While the aforementioned set of polynomials is S_n -stable, it does not form a basis for V_{n-1} in general. Berget and Rhoades work with a basis for V_{n-1} that is not S_n -stable. To establish that the restriction of V_{n-1} from S_n to S_{n-1} is indeed Haiman’s Park_{n-1} , Berget and Rhoades use Gröbner-theoretic techniques to construct a linear subspace of V_{n-1} with an S_{n-1} -stable monomial basis indexed by parking functions.

Our point of departure is a particular permutahedron in \mathbb{R}^n whose set of lattice points is equinumerous with the set of parking functions on length $n-1$, thereby providing a plausible candidate. Given a tuple $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$, we define the *permutahedron* $P_\lambda \subset \mathbb{R}^n$ to be the convex hull of the S_n orbit of λ . We denote the set of lattice points $P_\lambda \cap \mathbb{Z}^n$ by $\text{Lat}(P_\lambda)$. For $n \geq 2$, define δ_n to be the partition $(n-2, \dots, 1, 0, 0)$. It is clear that S_n acts on $\text{Lat}(P_{\delta_n})$. Let γ_n denote the associated representation. Here is our main result which answers the question posed by Berget and Rhoades.

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Theorem 1.1. *We have that*

$$\text{Res}_{S_{n-1}}^{S_n}(\gamma_n) = \text{Park}_{n-1}.$$

We refer the reader to Example 4.1 for the explicit computation in the case $n = 4$. Thus Theorem 1.1 tells us that γ_n is a permutation representation which extends the parking function representation. Furthermore, a conjecture of the first and third author [KT21, Conjecture 3.1] may be restated as claiming that γ_n is isomorphic to the ungraded Berget-Rhoades representation V_{n-1} .

Our approach is indirect and builds off of earlier work [KT21] by the first and third author wherein a family of S_n -representations $\widehat{\text{PF}}_{n,c}$ that restrict to Park_{n-1} is constructed. For an appropriately chosen value of c , this representation is isomorphic to what we consider here. We give an explicit h -positive expansion for the Frobenius characteristic of this representation in terms of binary Lyndon words satisfying a straightforward constraint. Our proof goes via an intermediate module $\mathcal{C}_{m,n}$ that we analyze in depth as well. The representation γ_n is obtained by identifying elements of $\mathcal{C}_{1,n}$ up to a natural equivalence relation. Finally, we can compute the character of γ_n by appealing to [KT21, Theorem 3.2]. This does not appear to be a straightforward task from the definition of γ_n . The character values allow us to make a connection with recent work of Ardila, Schindler and Vindas-Meléndez [ASVM20], which we state next.

Corollary 1.2. *Let Π_n denote the standard permutahedron $P_{(n-1, \dots, 1, 0)}$. Given $\sigma \in S_n$ with cycle type $(\lambda_1, \dots, \lambda_\ell)$, let Π_n^σ denote the set of points in Π_n that are fixed by σ . Suppose $\text{GCD}(\lambda_1, \dots, \lambda_\ell) = 1$. Then the normalized volume of Π_n^σ is equal to the number of lattice points in P_{δ_n} fixed by σ .*

If σ is the identity permutation, then Corollary 1.2 says that normalized volume of the standard permutahedron in \mathbb{R}^n is equal to the number of lattice points in P_{δ_n} . The former is well known to equal n^{n-2} [Sta91]. Thus, in this specific instance, our result reduces to a special case of [Pos09, Corollary 11.5].

For maximum generality, we work in the setting of rational parking functions for the majority of this paper. In Section 4, we specialize to arrive at Theorem 1.1.

2. THE SETUP

To keep our exposition brief, we refer the reader [Sta99, Chapter 7] for all notions pertaining to the combinatorics of symmetric functions which are not defined explicitly here.

2.1. The modules $\mathcal{C}_{m,n}$ and $\widehat{\mathcal{C}}_{m,n}$. Given positive integers m and n , set $N := mn$ and

$$c_{m,n} := \frac{(N-2)(n-1)}{2}.$$

This given, consider the set of N -tuples defined as follows:

$$\mathcal{C}_{m,n} := \{(x_1, \dots, x_N) \mid 0 \leq x_i \leq n-1, \sum_{1 \leq i \leq N} x_i = c_{m,n} \pmod{n}\}.$$

Clearly, $|\mathcal{C}_{m,n}| = n^{N-1}$. Geometrically, one may interpret $\mathcal{C}_{m,n}$ to be the set of lattice points in the cube $[0, n-1]^N$ in \mathbb{R}^N that lie on certain translates of the hyperplane $x_1 + \dots + x_N = 0$. Note that S_N acts on $\mathcal{C}_{m,n}$ by permuting coordinates and we denote the resulting permutation action by $\tau_{m,n}$. We abuse notation and use $\mathcal{C}_{m,n}$ to denote both the set and the resulting S_N -module.

Let Λ_k denote the set of tuples $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ in \mathbb{N}^k . Here \mathbb{N} denotes the set of nonnegative integers. We refer to elements of Λ_k as *partitions*. Given $(\lambda_1, \dots, \lambda_k) \in \Lambda_k$, we refer to λ_i 's as the *parts* of λ . In particular, we consider 0 to be a part. Given any sequence $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{N}^k$, we define $\text{sort}(\mathbf{x})$ to be the partition obtained by sorting \mathbf{x} in nonincreasing order. Let

$$Y_{m,n} := \Lambda_N \cap \mathcal{C}_{m,n}.$$

Clearly, elements of $Y_{m,n}$ index the orbits of $\mathcal{C}_{m,n}$ under $\tau_{m,n}$.

By drawing $\lambda \in Y_{m,n}$ as a Young diagram in French notation so that the lower left corner coincides with the origin in \mathbb{Z}^2 , we may identify λ with a lattice path L_λ that starts at $(n, 0)$, ends at $(0, N)$, and takes *vertical* and *horizontal* steps of unit length. All coordinates here are Cartesian. It will be convenient to extend L_λ to an infinite path L_λ^∞ by repeating L_λ . Figure 1 depicts $\lambda = (2, 2, 1, 1, 0) \in Y_{1,5}$. The shaded region represents the 5×5 box where λ is drawn, the red path depicts a fragment of L_λ^∞ , and the thickened subpath represents L_λ .

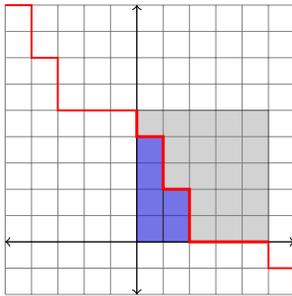


FIGURE 1. Path corresponding to $\lambda = (2, 2, 1, 1, 0)$.

We define the *shift* map shift mapping $\{0, \dots, n-1\}^N$ to itself via the rule

$$\text{shift}(x_1, \dots, x_N) := (x_1 + 1, \dots, x_N + 1),$$

where addition is performed modulo n . It is clear that shift^n is the identity map. This map allows us to define an equivalence relation \sim on $\mathcal{C}_{m,n}$ by declaring two sequences to be *equivalent* if one is obtained by applying shift^j to the other for some $j \in \mathbb{N}$. Since the sum of the coordinates remains invariant modulo n upon applying the shift map, our S_N action on $\mathcal{C}_{m,n}$ descends to an action $\widehat{\tau}_{m,n}$ on the set of equivalence classes $\mathcal{C}_{m,n}/\sim$. We denote this set (and the associated S_N module) by $\widehat{\mathcal{C}}_{m,n}$. Since every equivalence class has n elements, we have that $|\widehat{\mathcal{C}}_{m,n}| = n^{N-2}$.

Remark 2.1. The careful reader should note that our construction works equally well with $c_{m,n}$ replaced by any integer. Thus, one obtains a family of S_N modules in this manner. The analog of the set $\widehat{\mathcal{C}}_{m,n}$ has the property that every equivalence class therein has a unique element (x_1, \dots, x_N) so that (x_1, \dots, x_{N-1}) is a rational parking function. See discussion in [KT21, Section 5] to this end. It follows that the modules under consideration in this article are a special case of those studied in loc. cit., the choice $c_{m,n}$ having been made to answer the question of Berget and Rhoades. This choice is not merely a fortuitous coincidence as $c_{m,n}$ is closely related to the area statistic on parking functions, and the latter already plays a role in [BR14]. The one pertinent upshot of this discussion is that $\widehat{\mathcal{C}}_{1,n}$ restricts to Park_{n-1} .

2.2. Binary words and $Y_{m,n}$. We now interpret the partitions in $Y_{m,n}$ as certain words in the alphabet $\{0, 1\}$ as this will shed more light into their structure. By reading L_λ from right to left and recording a 0 (respectively 1) for each horizontal (respectively vertical) step, we obtain a word w_λ of length $(m+1)n$ in $\{0, 1\}$. Clearly, w_λ begins with a 0 and has mn 1s and n 0s. We refer to any word in the alphabet $\{0, 1\}$ with the property that the number of 1s is m times the number of 0s as *m -balanced*. We denote the length of a word w by $|w|$. For the partition $\lambda = (2, 2, 1, 1, 0) \in Y_{1,5}$ depicted in Figure 1, we have that $w_\lambda = 0001101101$.

We now associate a *weight* $\text{wt}(w)$ with any m -balanced word $w = w_1 \dots w_{(m+1)n}$:

$$\text{wt}(w) := \sum_{w_i=1} i.$$

This given, define

$$B_{m,n} := \{w \in \{0, 1\}^{(m+1)n} \mid w \text{ } m\text{-balanced}, w_1 = 0, \text{wt}(w) \equiv -1 \pmod{n}\}.$$

One may check that $w_{(2,2,1,1,0)} = 0001101101$ belongs to $B_{1,5}$ as its weight equals 34, which indeed is $-1 \pmod{5}$.

Before establishing a couple of lemmas that emphasize the importance of $B_{m,n}$, we need some notions from the combinatorics on words [Lot97]. Given a word $w = w_1 \dots w_k$, define $\text{rotate}(w) := w_2 \dots w_k w_1$. Clearly, $\text{rotate}^k(w) = w$. We say that w is *primitive* (or *aperiodic*) if no proper cyclic rotation of w coincides with w . In other words, $w = w_1 \dots w_k$ is primitive if $\text{rotate}^j(w) \neq w$ for $1 \leq j \leq k-1$. For instance, 0101 is not primitive while 0011 is. If w is not primitive, it may be written as $w = \widehat{w}^{k/d}$ for some primitive \widehat{w} and d a proper divisor of k (that is, d cannot equal k). We say that two words are *conjugate* if one is obtained as a cyclic rotation of the other. Observe that a conjugate of a primitive word is again primitive.

Lemma 2.2. *Every $w \in B_{m,n}$ is primitive.*

Proof. Towards establishing a contradiction, suppose w is not primitive. Then $w = \widehat{w}^{(m+1)n/d}$ where \widehat{w} is primitive and d is a proper divisor of $(m+1)n$. We claim that \widehat{w} is m -balanced as well. Indeed, say \widehat{w} possesses r 0s. It must be that

$$(2.1) \quad \frac{(m+1)n}{d} r = n,$$

which implies that $r = \frac{d}{m+1}$. Thus, the number of 1s in \widehat{w} is $\frac{md}{m+1}$, implying that \widehat{w} is m -balanced. Note in particular that $(m+1) \mid d$.

Suppose $\text{wt}(\widehat{w})$ equals M . Then we have that

$$(2.2) \quad \begin{aligned} \text{wt}(w) &= M + \left(M + \frac{md}{m+1}d\right) + \left(M + \frac{md}{m+1}2d\right) + \dots + \left(M + \frac{md}{m+1} \left(\frac{(m+1)n}{d} - 1\right)d\right) \\ &= \frac{(m+1)n}{d}M + \frac{md^2}{m+1} \binom{\frac{(m+1)n}{d}}{2}, \end{aligned}$$

which, modulo n , translates to the equality

$$(2.3) \quad \text{wt}(w) \equiv \frac{(m+1)n}{d}M - \frac{mnd}{2}.$$

Since $(m+1)|d$, we know that md is even. Thus (2.3) simplifies to

$$(2.4) \quad \text{wt}(w) = \frac{(m+1)n}{d}M \pmod{n}.$$

Since $w \in B_{m,n}$, we know that $\text{wt}(w) = -1 \pmod{n}$. This in conjunction with (2.4) implies that M satisfies

$$(2.5) \quad \frac{(m+1)n}{d}M = -1 \pmod{n}.$$

Writing $(m+1)n/d$ as $\frac{n}{d/(m+1)}$, and recalling that d is a proper divisor of $(m+1)n$, we conclude that $\text{GCD}(\frac{(m+1)n}{d}, n) \geq 2$. In particular, (2.5) has no solutions, and we have established that w is primitive. \square

We are now ready to relate $B_{m,n}$ to $Y_{m,n}$.

Lemma 2.3. *$B_{m,n}$ and $Y_{m,n}$ have the same cardinality.*

Proof. We claim that the correspondence $\lambda \mapsto w_\lambda$ is a bijection from $Y_{m,n}$ to $B_{m,n}$. To this end, we first show that $w_\lambda \in B_{m,n}$. For convenience, set $w = w_1 \dots w_{(m+1)n} := w_\lambda$. It is immediate that $w_1 = 0$ as $\lambda_1 \leq n-1$, and that w is m -balanced. Thus we need to check that $\text{wt}(w) = -1 \pmod{n}$. It is easy to see that $w_j = 1$ if and only if $j = n - \lambda_i + i$ for some (unique) i . Thus we obtain

$$(2.6) \quad \text{wt}(w) = \sum_{1 \leq i \leq N} (n - \lambda_i + i) = Nn + \binom{N+1}{2} - |\lambda|.$$

Since $N = mn$ and $|\lambda| = (N-2)(n-1)/2$, we have

$$(2.7) \quad \text{wt}(w) = Nn + \binom{m}{2}n^2 + mn + n - 1.$$

Thus we conclude that $\text{wt}(w) = -1 \pmod{n}$, and therefore $w \in B_{m,n}$.

It is clear that this correspondence is an injection from $Y_{m,n}$ to $B_{m,n}$. That this is a bijection follows because this correspondence is easily reversible, and one may obtain a partition for every word in $B_{m,n}$. That this partition belongs to $Y_{m,n}$ follows by reading the earlier string of equalities backwards. \square

We use this correspondence to obtain a ‘closed form’ for $|Y_{m,n}|$.

Corollary 2.4. *The number of orbits of $\mathcal{C}_{m,n}$ under $\tau_{m,n}$ equals the number of N -element subsets of $[(m+1)n-1] := \{1, \dots, (m+1)n-1\}$ whose subset sum equals $-1 \pmod{n}$. More explicitly, we have*

$$|Y_{m,n}| = \frac{1}{n} \sum_{d|n} (-1)^{m(n+d)} \mu(n/d) \binom{(m+1)d-1}{md}.$$

Proof. Given $w = w_1 \dots w_{(m+1)n} \in B_{m,n}$, associate an N -element subset S_w of $[(m+1)n-1]$ by

$$S_w = \{j-1 \mid w_j = 1\}.$$

It is clear that the sum of elements in S_w is $-1 \pmod{n}$, and that this correspondence sets up a bijection between $B_{m,n}$ and N -element subsets of $[(m+1)n-1]$ with subset sum equal to

$-1 \pmod{n}$. It remains to count such subsets, and we appeal to [Che19, Theorem 1.1] to this end. Setting $u = m + 1$, $v = m$ and $c = n - 1$ in loc. cit. implies that

$$(2.8) \quad \begin{aligned} B_{m,n} &= \frac{1}{(m+1)n} \sum_{d|n} (-1)^{m(n+d)} \mu(n/d) \binom{(m+1)d}{md} \\ &= \frac{1}{n} \sum_{d|n} (-1)^{m(n+d)} \mu(n/d) \binom{(m+1)d-1}{md}, \end{aligned}$$

which completes the proof. \square

3. THE h -POSITIVITY OF $\text{Frob}(\widehat{\tau}_{m,n})$

In this section we establish the h -positivity of the action $\widehat{\tau}_{m,n}$ on $\widehat{\mathcal{C}}_{m,n}$. We do this by determining representatives of equivalence classes in $\widehat{\mathcal{C}}_{m,n}$ that carry the natural action of S_N which permutes coordinates. As remarked earlier, in [KT21], we made a choice of representatives with the property that the first $N - 1$ coordinates give a rational parking function. Unfortunately, this choice does not give rise to an S_N -stable set typically.

Lemma 3.1. *For every $\lambda \in Y_{m,n}$, the cardinality of the set $\{\text{sort} \circ \text{shift}^j(\lambda) \mid 0 \leq j \leq n - 1\}$ is n .*

Proof. We exploit a nice connection between rotations of m -balanced words and the shift map applied to elements of $Y_{m,n}$. Consider the lattice path L_λ associated to $\lambda \in Y_{m,n}$, as well as its infinite extension L_λ^∞ . Label the $(m+1)n$ steps of L_λ with integers 1 through $(m+1)n$ going right to left.

Let the labels of the horizontal steps be $a_0 < a_1 < \dots < a_{n-1}$. This given, here is how one may compute $\text{sort} \circ \text{shift}^j(\lambda)$ for $0 \leq j \leq n - 1$. Consider the subpath of L_λ^∞ of length $(m+1)n$ that begins with the horizontal step labeled a_j and proceeds northwest. This subpath may be treated as $L_{\widetilde{\lambda}}$ (after a potential translation) for a unique partition $\widetilde{\lambda} \in Y_{m,n}$. It is not hard to see that $\widetilde{\lambda} = \text{sort} \circ \text{shift}^j(\lambda)$ by realizing that $\text{sort} \circ \text{shift}$ is essentially a ‘rotation’ given that λ is weakly decreasing from left to right. For instance, consider Figure 2 where the partition $\text{sort} \circ \text{shift}^3(\lambda)$ is depicted by the parallel-hatched region in the translated 5×5 box for $\lambda = (2, 2, 1, 1, 0)$.

From the fact that L_λ^∞ is built by periodically repeating L_λ , it follows that

$$(3.1) \quad w_{\widetilde{\lambda}} = \text{rotate}^{a_j-1}(w_\lambda).$$

Now, recall that w_λ is primitive by Lemma 2.2, and thus $w_{\widetilde{\lambda}}$ and w_λ are distinct. This in turn implies that $\widetilde{\lambda}$ and λ are distinct as well. It follows that the set $\{\text{sort} \circ \text{shift}^j(\lambda) \mid 0 \leq j \leq n - 1\}$ indeed has n elements, one for each conjugate of w_λ beginning with a 0. \square

We now exploit this lemma to extract a set of representatives for the equivalence classes in $\widehat{\mathcal{C}}_{m,n}$ that is S_N -stable.

Given $w \in B_{m,n}$, we denote the associated partition λ satisfying $w_\lambda = w$ as λ_w . Recall that the conjugacy class of w consists of all $(m+1)n$ cyclic rotations thereof. Amongst these cyclic rotations, there is a unique lexicographically smallest word, where the order is inherited by declaring $0 < 1$. Such a word is known as a *Lyndon word*. Observe the crucial fact that Lyndon word in the conjugacy

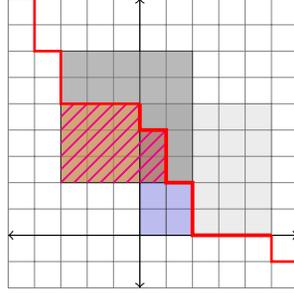


FIGURE 2. The partition $\text{sort} \circ \text{shift}^3(\lambda)$ realized by $L_{\tilde{\lambda}}$ where $\lambda = (2, 2, 1, 1, 0)$.

class of $w \in B_{m,n}$ must itself belong to $B_{m,n}$, as it must begin with a 0 and cyclic rotations preserve weights. We denote the set of Lyndon words in $B_{m,n}$ by $B_{m,n}^L$.

Example 3.2. Consider $\mathcal{C}_{1,4}$. The 8 partitions in $Y_{1,4}$ (commas and parentheses suppressed) are given below. Those in the same column are obtained by applying $\text{sort} \circ \text{shift}^j$ for $0 \leq j \leq 3$ to the highlighted partition.

$$\begin{array}{cc} 2100 & 1110 \\ 3211 & 2221 \\ 3220 & 3332 \\ 3310 & 3000 \end{array}$$

It can be checked that the words w_λ corresponding to the highlighted partitions are indeed Lyndon, and thus

$$B_{1,4}^L = \{00101011, 00011101\}.$$

More importantly, since elements of $Y_{1,4}$ index S_4 -orbits of $\mathcal{C}_{1,4}$, and $\widehat{\mathcal{C}}_{1,4}$ is obtained by identifying elements of $\mathcal{C}_{1,4}$ up to shifts, it follows that the S_4 -orbits of 2100 and 1110 generate a system of representatives for equivalence classes in $\widehat{\mathcal{C}}_{1,4}$. This is the underlying idea of what follows.

Let \mathcal{O}_{λ_w} denote the S_N -orbit of λ_w for $w \in B_{m,n}$. We claim that the set of elements of $\mathcal{C}_{m,n}$ that belong to the orbit of λ_w for a Lyndon word $w \in B_{m,n}$ gives a complete set of representatives for equivalence classes in $\widehat{\mathcal{C}}_{m,n}$. Indeed, we know that

$$(3.2) \quad \mathcal{C}_{m,n} = \coprod_{w \in B_{m,n}} \mathcal{O}_{\lambda_w},$$

and by invoking Lemma 3.1 to rewrite the right-hand side of (3.2) we get that

$$\begin{aligned} \mathcal{C}_{m,n} &= \coprod_{w \in B_{m,n}^L} \coprod_{0 \leq j \leq n-1} \mathcal{O}_{\text{sort} \circ \text{shift}^j(\lambda_w)} \\ (3.3) \quad &= \coprod_{w \in B_{m,n}^L} \coprod_{0 \leq j \leq n-1} \mathcal{O}_{\text{shift}^j(\lambda_w)}. \end{aligned}$$

Since shift commutes with the action of S_N , we can rewrite (3.3) as

$$(3.4) \quad \mathcal{C}_{m,n} = \coprod_{w \in B_{m,n}^L} \coprod_{0 \leq j \leq n-1} \text{shift}^j(\mathcal{O}_{\lambda_w}),$$

where we interpret $\text{shift}^j(\mathcal{O}_{\lambda_w})$ as the set obtained by applying shift^j to all elements in \mathcal{O}_{λ_w} . Since $\widehat{\mathcal{C}}_{m,n}$ is obtained by identifying sequences in $\mathcal{C}_{m,n}$ up to shifts, (3.4) tells us that we may identify $\widehat{\mathcal{C}}_{m,n}$ with $\coprod_{w \in B_{m,n}^L} \mathcal{O}_{\lambda_w}$, and thus $\widehat{\tau}_{m,n}$ is indeed the permutation action on the latter set. We are now ready to record an immediate consequence of this argument.

Theorem 3.3. *To each $w \in B_{m,n}^L$, associate a sequence $c(w)$ that records the lengths of the maximal runs of 1s in w . Then we have that*

$$\text{Frob}(\widehat{\tau}_{m,n}) = \sum_{w \in B_{m,n}^L} h_{\text{sort}(c(w))}.$$

In particular, we have that the number of orbits of $\widehat{\mathcal{C}}_{m,n}$ under $\widehat{\tau}_{m,n}$ is given by

$$|B_{m,n}^L| = \frac{|Y_{m,n}|}{n} = \frac{1}{n^2} \sum_{d|n} (-1)^{m(n+d)} \mu(n/d) \binom{(m+1)d-1}{md}.$$

Recall that in [KT21, Theorem 6.1], the number of orbits was computed by way of explicit character values. See also [Ray18, Section 5] for a topological interpretation for the numbers $|B_{m,n}^L|$.

Example 3.4. Consider $m = 2$ and $n = 3$. The three Lyndon words in $B_{m,n}^L$ are

$$001011111 \quad 001111011 \quad 010110111.$$

The corresponding $\text{sort}(c(w))$ are 51, 42, and 321. Theorem 3.3 implies that

$$\text{Frob}(\widehat{\tau}_{m,n}) = h_{51} + h_{42} + h_{321}.$$

4. THE TRIMMED STANDARD PERMUTAHEDRON

In this section we focus on the case $m = 1$, or equivalently, $N = n$. Hence we suppress the m from all notions introduced earlier. Our goal is to establish Theorem 1.1 stated in the introduction.

Recall that given $\lambda := (\lambda_1 \geq \dots \geq \lambda_n) \in \mathbb{Z}^n$, we let P_λ denote the polytope in \mathbb{R}^n defined by considering the convex hull of the S_n orbit of λ . The P_λ 's are referred to as *usual permutahedra*. The set of lattice points in P_λ , that is $\text{Lat}(P_\lambda)$, is clearly S_n -stable, and one obtains a natural class of S_n -modules in this manner. Furthermore, since the stabilizer of any point in $\text{Lat}(P_\lambda)$ is a Young subgroup of S_n , we are guaranteed h -positivity of the associated Frobenius characteristic. It is a priori unclear whether these modules are of any value other than the intrinsic one. In what follows, we discuss the case of a special permutahedron, and show that its set of lattice points indexes the orbits of $\widehat{\mathcal{C}}_n$ under the action of $\widehat{\tau}_n$.

Fix $n \geq 2$. Let $\delta_n = (n-2, \dots, 1, 0, 0) \in \mathbb{N}^n$. We are interested in the S_n action on $\text{Lat}(P_{\delta_n})$, which we denote by γ_n . The reason behind this is the equality

$$(4.1) \quad |\text{Lat}(P_{\delta_n})| = n^{n-2}.$$

We briefly explain how to arrive at this equality through work of Postnikov [Pos09], leaving it to the reader to check loc. cit. for further details. Consider the *standard permutahedron* $P_{(n-1, \dots, 1, 0)}$. The Minkowski difference of the standard permutahedron with the standard simplex $P_{(1, 0, \dots, 0)}$ is another permutahedron $P_{(n-2, n-2, \dots, 1, 0)}$. Following [Pos09, Definition 11.2], we refer to the latter as the *trimmed standard permutahedron*. By [Pos09, Corollary 11.5], we have that

$$(4.2) \quad |\text{Lat}(P_{(n-2, n-2, \dots, 1, 0)})| = n^{n-2}.$$

That this is equivalent to the equality in (4.1) is because translating $P_{(n-2, n-2, \dots, 1, 0)}$ by $(n-2, \dots, n-2)$ followed by negating all coordinates maps it to P_{δ_n} . It is also clear that this map is S_n -equivariant, so the S_n action on P_{δ_n} is isomorphic to that on $P_{(n-2, n-2, \dots, 1, 0)}$, which also explains the title of this section.

The right-hand side of (4.1) naturally raises the question whether this S_n -action is related to the parking function representation. Indeed, as we shall soon establish, upon restricting this action to S_{n-1} we recover the parking function representation.

Example 4.1. Suppose $n = 4$. Then $\delta_n = (2, 1, 0, 0)$. The 16 elements in $\text{Lat}(P_{\delta_n})$ are given by the S_4 -orbits of $(2, 1, 0, 0)$ and $(1, 1, 1, 0)$; compare these two orbit representatives with those encountered in Example 3.2. It follows that

$$\text{Frob}(\gamma_4) = h_{211} + h_{31},$$

which upon restricting to S_3 gives $h_3 + 3h_{21} + h_{111}$. This last expression is the Frobenius characteristic of the S_3 action on parking functions of length 3. An alternative perspective is by projecting P_{δ_4} onto the hyperplane $x_4 = 0$ in \mathbb{R}^4 and realizing that S_3 acts on the lattice points of the resulting polytope. The h_{111} term comes from the orbit of the point $(2, 1, 0)$, the three h_{21} terms come from the orbits of $(1, 0, 0)$, $(2, 0, 0)$, and $(1, 1, 0)$ respectively, and the h_3 term comes from the orbit of $(1, 1, 1)$.

To establish that γ_n is isomorphic to τ_n , we identify representatives of the equivalence classes in $\widehat{\mathcal{C}}_n$ that belong to $\text{Lat}(P_{\delta_n})$. Clearly, the S_n action on $\text{Lat}(P_{\delta_n})$ has orbits indexed by elements of $\text{Par}_{\leq \delta_n}$, which we defined to be the set of lattice points $(\lambda_1, \dots, \lambda_n)$ in P_{δ_n} such that $\lambda_1 \geq \dots \geq \lambda_n$. Put differently, $\text{Par}_{\leq \delta_n}$ consists of all partitions of size $\binom{n-1}{2}$ and length at most n that are dominated by δ_n [Rad52]. Note that all elements in $\text{Par}_{\leq \delta_n}$ do indeed belong to \mathcal{C}_n .

Lemma 4.2. *Given $\lambda = (\lambda_1, \dots, \lambda_n) \in \text{Par}_{\leq \delta_n}$, no element in the set $\{\text{sort} \circ \text{shift}^j(\lambda) \mid 1 \leq j \leq n-1\}$ belongs to $\text{Par}_{\leq \delta_n}$.*

Proof. We employ the lattice paths L_λ and L_λ^∞ defined earlier. Label the horizontal steps in L_λ with integers 0 through $n-1$ going right to left. Fix a j such that $0 \leq j \leq n-1$. Consider the fragment L' of L_λ^∞ of length $2n$ that starts with the horizontal step labeled j and proceeds northwest. As discussed before, the partition determined by L' (in the bottom left corner of the $n \times n$ box this path naturally lives in) is $\text{sort} \circ \text{shift}^j(\lambda)$. If we let i denote the number of vertical steps in L preceding the horizontal step labeled j , then we have

$$(4.3) \quad |\text{sort} \circ \text{shift}^j(\lambda)| = |\lambda| + (j-i)n.$$

Now suppose there exists $j \neq 0$ such that $|\text{sort} \circ \text{shift}^j(\lambda)| \in \text{Par}_{\leq \delta_n}$. From (4.3) it follows that $j = i$. Thus, the horizontal step labeled j must touch the diagonal $x+y = n$. Let $\nu = (\nu_1, \dots, \nu_{n-j})$

be the partition determined by the subpath of L restricted to the $(n-j) \times (n-j)$ box in the top left. Let $\mu = (\mu_1, \dots, \mu_j)$ be the partition determined by the subpath of L restricted to the $j \times j$ box in the bottom right. Now, observe that

$$(4.4) \quad \lambda = (n-j + \mu_1, n-j + \mu_2, \dots, n-j + \mu_j, \nu_1, \dots, \nu_{n-j})$$

$$(4.5) \quad \text{sort} \circ \text{shift}^j(\lambda) = (j + \nu_1, j + \nu_2, \dots, j + \nu_{n-j}, \mu_1, \dots, \mu_j).$$

Since λ is dominated by δ_n , we know that

$$(4.6) \quad \sum_{k=1}^j (n-j + \mu_k) \leq \sum_{k=1}^j (n-j + k - 2) = j(n-j-1) + \binom{j}{2}.$$

Since the left-hand side is $|\lambda| - |\nu| = \binom{n-1}{2} - |\nu|$, we may rewrite the above inequality as

$$(4.7) \quad |\nu| \geq \binom{n-1}{2} - j(n-j-1) - \binom{j}{2} = \binom{n-j-1}{2}.$$

On the other hand, since our assumption is that $\text{sort} \circ \text{shift}^j(\lambda)$ is also dominated by δ_n , we obtain

$$(4.8) \quad \sum_{k=1}^{n-j} (j + \nu_k) \leq \sum_{k=1}^{n-j} (j-2+k) = (n-j)(j-1) + \binom{n-j}{2}.$$

This in turn may be rewritten as

$$(4.9) \quad |\nu| \leq \binom{n-j}{2} - n + j = \binom{n-j-1}{2} - 1,$$

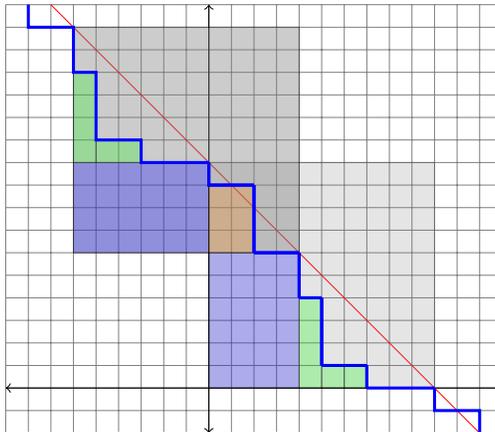
which is in contradiction with the inequality in (4.7). \square

To illustrate, consider $n = 10$ and $\lambda = (7, 5, 5, 5, 4, 4, 2, 2, 2, 0) \vdash 36$ as shown in Figure 3. The path L_λ is shown in the 10×10 box shaded light gray. The red line shows the diagonal $y + x = 10$ and a horizontal step touching this diagonal gives a value j such that $\text{sort} \circ \text{shift}^j(\lambda) \vdash 36$. In the figure, we have chosen $j = 6$. The path L' is shown in the 10×10 box with the darker shade of gray, and we see that $\text{sort} \circ \text{shift}^6(\lambda) = (8, 8, 8, 6, 3, 1, 1, 1, 0, 0)$. Note that $\nu = (2, 2, 2, 0)$ is determined by the path in the intersection of the two shaded regions. Also $\mu = (3, 1, 1, 1, 0, 0)$ is the partition determined by subpath of L_λ by the 6×6 box in the bottom right. The reader may verify that $(8, 8, 8, 6, 3, 1, 1, 1, 0, 0) \notin \text{Par}_{\leq \delta_{10}}$.

We are now ready to establish our central result which essentially says that elements of $\text{Lat}(P_{\delta_n})$ form a complete set of representatives for equivalence classes in $\widehat{\mathcal{C}}_n$.

Theorem 4.3. *The representation γ_n obtained by the S_n action on $\text{Lat}(P_{\delta_n})$ is isomorphic to the representation $\widehat{\tau}_n$ obtained by the S_n action on $\widehat{\mathcal{C}}_n$. Furthermore, the restriction of γ_n to S_{n-1} is Park_{n-1} . The explicit h -expansion of $\text{Frob}(\gamma_n)$ may be obtained as follows: Suppose $\text{mult}(\lambda)$ denotes the partition recording the multiplicities of each part in λ for $\lambda \in \text{Par}_{\leq \delta_n}$ (recall we are allowing 0 to be a part as well). Then*

$$\text{Frob}(\gamma_n) = \text{Frob}(\widehat{\tau}_n) = \sum_{\lambda \in \text{Par}_{\leq \delta_n}} h_{\text{mult}(\lambda)}.$$

FIGURE 3. Path corresponding to $\lambda = (7, 5, 5, 5, 4, 4, 2, 2, 2, 0)$.

Proof. Note that each element in $\text{Lat}(P_{\delta_n})$ belongs to \mathcal{C}_n , and hence is a representative of some equivalence class in $\widehat{\mathcal{C}}_n$. Define $\varphi : \text{Lat}(P_{\delta_n}) \rightarrow \widehat{\mathcal{C}}_n$ sending an element to the equivalence class in $\widehat{\mathcal{C}}_n$ that contains it. We claim that φ is an S_n -equivariant bijection. The S_n -equivariance is immediate and so we focus on establishing bijectivity. Since $|\text{Lat}(P_{\delta_n})| = |\widehat{\mathcal{C}}_n| = n^{n-2}$, it suffices to show that φ is an injection.

For $\lambda \in \text{Par}_{\leq \delta_n}$, let \mathcal{O}_λ denote the S_n -orbit of λ . Then we know that

$$(4.10) \quad \text{Lat}(P_{\delta_n}) = \coprod_{\lambda \in \text{Par}_{\leq \delta_n}} \mathcal{O}_\lambda.$$

By Lemma 4.2, we infer that for distinct partitions γ and μ in $\text{Par}_{\leq \delta_n}$, we have $\varphi(\gamma) \neq \varphi(\mu)$. We now show that for distinct elements x and y in \mathcal{O}_λ where $\lambda \in \text{Par}_{\delta_n}$ is fixed, we have that $\varphi(x) \neq \varphi(y)$. Indeed suppose for the sake of contradiction that $\varphi(x) = \varphi(y)$. Since the S_n action on \mathcal{C}_n commutes with the shift map, we may assume without loss of generality that one of our two chosen elements, say x , equals λ . Suppose $y = \sigma \cdot \lambda$ for some $\sigma \in S_n$ that does not equal the identity. Now our hypothesis says that $\varphi(\lambda) = \varphi(\sigma \cdot \lambda)$, and hence we must have that $\sigma \cdot \lambda = \text{shift}^j(\lambda)$ for some $1 \leq j \leq n-1$. But then $\text{sort} \circ \text{shift}^j(\lambda) = \lambda \in \text{Par}_{\leq \delta_n}$, which is a contradiction in view of Lemma 4.2. Finally we note that a similar argument tells us that if pick $x \in \mathcal{O}_\lambda$ and $y \in \mathcal{O}_\mu$ for $\lambda \neq \mu$ in Par_{δ_n} , then $\varphi(x) \neq \varphi(y)$.

In view of these facts just established and equation (4.10), we conclude that φ is injective, and hence bijective. This discussion in conjunction with Remark 2.1 yields the statement of the theorem. \square

Taking [KT21, Theorem 3.1] into account, we get the following result.

Corollary 4.4. *Let $\pi \in S_n$ have cycle type $\lambda = (\lambda_1, \dots, \lambda_\ell)$ where $\lambda_\ell > 0$. Set $d := \text{GCD}(\lambda_1, \dots, \lambda_\ell)$. Then the number of lattice points in $\text{Lat}(P_{\delta_n})$ fixed by the action of π is given by $f(d)n^{\ell-2}$ where*

$$f(d) = \begin{cases} 1 & d = 1, \\ 2 & d = 2 \text{ and } n = 2 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that Corollary 1.2 now follows in view of [ASVM20, Theorem 1.2]. Is there a combinatorial proof of Corollary 4.4 which eschews character-theoretic computations?

We conclude with a couple of remarks.

Remark 4.5. In private communication with the authors, S. Backman informed them that lattice points in the trimmed permutahedron P_{δ_n} may be interpreted as *break divisors* on the complete graph K_n , and the latter are in bijection with the set of spanning trees of K_n . Furthermore, in the divisor group of K_n , break divisors are in the same equivalence class as *q-reduced divisors*, which turn out to be usual parking functions. We refer the reader to [BN07, ABKS14, Bac17] for more on these beautiful connections. By appealing to these (non-trivial) results, one could have bypassed our elementary Lemma 4.2 to arrive at Theorem 4.3.

Remark 4.6. One could ask for a generalization of Theorem 4.3 when $m > 1$. The following example shows that a naïve generalization may not work. Consider $m = 2$ and $n = 3$. Thus $c_{m,n} = 1 \pmod{3}$. The orbits of $\mathcal{C}_{m,n}$ are indexed by the partitions

100000	111100	211000
211111	222211	221110
222220	220000	222100

where the partitions in each column are obtained by applying $\text{sort} \circ \text{shift}^j$ to the partition in the top row. Any three partitions, one from each column, index S_6 -orbits for the action on $\widehat{\mathcal{C}}_{2,3}$. It is clear that in this instance that there is no way to pick three such partitions, all of the same size.

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REFERENCES

- [ABKS14] Yang An, Matthew Baker, Greg Kuperberg, and Farbod Shokrieh. Canonical representatives for divisor classes on tropical curves and the matrix-tree theorem. *Forum Math. Sigma*, 2:e24, 25, 2014. 12
- [ASVM20] Federico Ardila, Anna Schindler, and Andrés R. Vindas-Meléndez. The equivariant volumes of the permutahedron. *Sém. Lothar. Combin.*, 82B:Art. 16, 12, 2020. 2, 12
- [Bac17] Spencer Backman. Riemann-Roch theory for graph orientations. *Adv. Math.*, 309:655–691, 2017. 12
- [BN07] Matthew Baker and Serguei Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007. 12
- [BR14] Andrew Berget and Brendon Rhoades. Extending the parking space. *J. Combin. Theory Ser. A*, 123:43–56, 2014. 1, 3
- [Che19] Shane Chern. An extension of a formula of Jovovic. *Integers*, 19:Paper No. A47, 7, 2019. 6
- [dG10] Giacomo d’Antonio and Giovanni Gaiffi. Symmetric group actions on the cohomology of configurations in \mathbb{R}^d . *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 21(3):235–250, 2010. 1
- [ER19] Nicholas Early and Victor Reiner. On configuration spaces and Whitehouse’s lifts of the Eulerian representations. *J. Pure Appl. Algebra*, 223(10):4524–4535, 2019. 1

- [Gai96] Giovanni Gaiffi. The actions of S_{n+1} and S_n on the cohomology ring of a Coxeter arrangement of type A_{n-1} . *Manuscripta Math.*, 91(1):83–94, 1996. 1
- [Hai94] Mark D. Haiman. Conjectures on the quotient ring by diagonal invariants. *J. Algebraic Combin.*, 3(1):17–76, 1994. 1
- [KT21] Matjaž Konvalinka and Vasu Tewari. Some natural extensions of the parking space. *J. Combin. Theory Ser. A*, 180:105394, 2021. 2, 3, 6, 8, 11
- [Lot97] M. Lothaire. *Combinatorics on words*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1997. With a foreword by Roger Lyndon and a preface by Dominique Perrin, Corrected reprint of the 1983 original, with a new preface by Perrin. 4
- [Mat96] Olivier Mathieu. Hidden Σ_{n+1} -actions. *Comm. Math. Phys.*, 176(2):467–474, 1996. 1
- [Pos09] A. Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009. 2, 9
- [PS04] A. Postnikov and B. Shapiro. Trees, parking functions, syzygies, and deformations of monomial ideals. *Trans. Amer. Math. Soc.*, 356(8):3109–3142, 2004. 1
- [Rad52] R. Rado. An inequality. *J. London Math. Soc.*, 27:1–6, 1952. 9
- [Ray18] Steven Rayan. Aspects of the topology and combinatorics of Higgs bundle moduli spaces. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 14:Paper No. 129, 18, 2018. 8
- [Sta91] Richard P. Stanley. A zonotope associated with graphical degree sequences. In *Applied geometry and discrete mathematics*, volume 4 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 555–570. Amer. Math. Soc., Providence, RI, 1991. 2
- [Sta99] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. 2
- [Whi97] Sarah Whitehouse. The Eulerian representations of Σ_n as restrictions of representations of Σ_{n+1} . *J. Pure Appl. Algebra*, 115(3):309–320, 1997. 1

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