

# NONCOMMUTATIVE LR COEFFICIENTS AND CRYSTAL REFLECTION OPERATORS

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ABSTRACT. We relate noncommutative Littlewood-Richardson coefficients of Bessenrodt-Luoto-van Willigenburg to classical Littlewood-Richardson coefficients via crystal reflection operators. A key role is played by the combinatorics of frank words.

## 1. INTRODUCTION

Quasisymmetric Schur functions, introduced by Haglund-Luoto-Mason-van Willigenburg [15], form a prominent basis for the Hopf algebra of quasisymmetric functions  $\text{QSym}$ . The quasisymmetric Schur function indexed by the composition  $\alpha$ , denoted by  $\mathcal{S}_\alpha$ , is obtained by summing monomials attached to semistandard composition tableaux of shape  $\alpha$ . This is reminiscent of the definition of Schur functions as sums of monomials corresponding to semistandard Young tableaux. As the name suggests, quasisymmetric Schur functions share many properties with classical Schur functions, and Mason's map  $\rho$  [27] connects the combinatorics of composition tableaux to that of Young tableaux. Understanding analogues of Schur functions and their generalizations has long been a theme in algebraic combinatorics; see [1, 2, 3, 4, 5, 28, 31, 36] for recent work in this context.

The Hopf algebra of noncommutative symmetric functions  $\text{NSym}$ , introduced in the seminal paper [13], is Hopf-dual to  $\text{QSym}$  as shown by Malvenuto-Reutenauer [26]. Bessenrodt-Luoto-van Willigenburg [6] studied the dual basis elements  $\mathbf{s}_\alpha$  corresponding to quasisymmetric Schur functions. The resulting functions, also indexed by compositions, are called noncommutative Schur functions. The inclusion of the Hopf algebra of symmetric functions  $\text{Sym}$  into  $\text{QSym}$  induces a projection  $\chi : \text{NSym} \twoheadrightarrow \text{Sym}$ . This projection maps noncommutative Schur functions to classical Schur functions, and justifies the name of the former. The structure constants  $C_{\alpha\beta}^\gamma$  that arise in

$$(1.1) \quad \mathbf{s}_\alpha \cdot \mathbf{s}_\beta = \sum_{\gamma} C_{\alpha\beta}^\gamma \mathbf{s}_\gamma$$

are called noncommutative Littlewood-Richardson (LR) coefficients. These coefficients turn out to be nonnegative integers that refine the classical LR coefficients  $c_{\nu\mu}^\lambda$  that arise in the

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product of Schur functions

$$(1.2) \quad s_\nu \cdot s_\mu = \sum_{\lambda} c_{\nu\mu}^\lambda s_\lambda.$$

More precisely, suppose  $\nu$  and  $\mu$  are partitions that rearrange to compositions  $\alpha$  and  $\beta$  respectively. Applying  $\chi$  to both sides in (1.1) and comparing the result with (1.2) implies

$$(1.3) \quad c_{\nu\mu}^\lambda = \sum_{\gamma} C_{\alpha\beta}^\gamma$$

where the sum on the right runs over all compositions  $\gamma$  that rearrange to  $\lambda$ . Among the numerous combinatorial interpretations of  $c_{\nu\mu}^\lambda$ , the one we focus on states that  $c_{\nu\mu}^\lambda$  counts LR tableaux of shape  $\lambda/\mu$  and content  $\nu$ . Our primary goal in this article is to describe the summands on the right hand side in (1.3) in terms of LR tableaux. To this end, crystal reflection operators are key.

To state our result, we introduce the necessary notation briefly. The reader is referred to Section 2 for details. Given a composition  $\alpha$ , we denote the partition underlying  $\alpha$  by  $\text{sort}(\alpha)$ . Let  $\text{LRT}(\lambda, \mu, \nu)$  be the set of LR tableaux of shape  $\lambda/\mu$  and content  $\nu$ . Given a permutation  $\sigma$ , let  $\text{LRT}^\sigma(\lambda, \mu, \nu)$  be the set of tableaux obtained by applying the crystal reflection operators corresponding to a reduced word of  $\sigma$ . Let  $\alpha := \sigma \cdot \nu$ , and let  $\beta$  be any composition that satisfies  $\text{sort}(\beta) = \mu$ . On applying the map  $\rho_\beta^{-1}$  (which sends Young tableaux to composition tableaux, [24, Chapter 4]) to elements in  $\text{LRT}^\sigma(\lambda, \mu, \nu)$ , we obtain the disjoint decomposition

$$(1.4) \quad \text{LRT}^\sigma(\lambda, \mu, \nu) = \coprod_{\gamma} X_{\alpha\beta}^\gamma,$$

where  $X_{\alpha\beta}^\gamma$  consists of all tableaux  $T \in \text{LRT}^\sigma(\lambda, \mu, \nu)$  whose outer shape is given by the composition  $\gamma$  under the map  $\rho_\beta^{-1}$ . Under this setup, our main theorem states the following.

**Theorem 1.1.** *The noncommutative LR coefficient  $C_{\alpha\beta}^\gamma$  equals the cardinality of  $X_{\alpha\beta}^\gamma$ .*

The upshot of Theorem 1.1 is that starting from  $\text{LRT}(\lambda, \mu, \nu)$ , we can compute *all* noncommutative LR coefficients  $C_{\alpha\beta}^\gamma$ , where  $\alpha$  dictates the choice of crystal reflections to be performed and  $\beta$  determines the generalized  $\rho$  map to be applied. We also interpret Theorem 1.1 in terms of chains in Young's lattice indexed by certain frank words and obtain a rule for  $C_{\alpha\beta}^\gamma$  involving box-adding operators on compositions. In the case where  $\alpha$  is either a partition or reverse partition, our interpretations yield the two LR rules in [7]. For an example demonstrating Theorem 1.1, see Subsection 3.2.

Note that LR coefficients can be interpreted as the number of lattice points in an appropriate convex polytope/cone [29]. Since Theorem 1.1 describes noncommutative LR coefficients by refining the set of LR tableaux, it is natural to ask if noncommutative LR coefficients count lattice points in some refinement of the corresponding polytope/cone. Data suggest that the analogue of the saturation conjecture, famously resolved by Knutson and Tao [16],

holds for noncommutative LR coefficients as well. In fact (1.3) tells us that saturation of noncommutative LR coefficients implies that of the classical counterparts. This motivates studying these constants from a polytopal perspective, which is why the authors were interested in a statement like Theorem 1.1 in the first place.

**Outline of the article:** Section 2 sets up all the necessary combinatorial background. Section 3 describes our central result along with examples. Section 4.3 identifies LR tableaux to certain distinguished frank words, called compatible words, drawing upon work by Remmel-Shimozono [33]. In Section 4.4, we introduce frank words and describe the Lascoux-Schützenberger symmetric group action on frank words. Section 4.5 relates crystal reflections acting on LR tableaux to the symmetric group action on compatible words. Section 5 relates the results in Section 4 back to noncommutative LR coefficients. We conclude our article with Corollary 5.4, which reinterprets our combinatorial interpretation for noncommutative LR coefficients in terms of box-adding operators on compositions.

## 2. BACKGROUND

To keep this section brief, we assume knowledge of combinatorial structures and algorithms arising in the well-studied theory of symmetric functions such as partitions, skew shapes, Young tableaux, Robinson-Schensted-Knuth insertion and jeu-de-taquin. The reader is referred to standard texts [12, 25, 34, 37] for further information.

Regarding notions relevant to the theory of quasisymmetric Schur functions, we adhere to the notation and conventions in [24]. *We emphasize here that what we call quasisymmetric Schur functions in this article are the Young quasisymmetric Schur functions in [24].* This choice, although cosmetic as both the functions are related via a simple involution, has its benefits. The tableau objects that we consider in the quasisymmetric/noncommutative setting align with the more prevalent notion of Young tableaux rather than reverse tableaux. Furthermore, it is the Young quasisymmetric Schurs that decompose the dual immaculate quasisymmetric functions [5] positively as shown recently in [1], thereby connecting two well-studied quasisymmetric analogues of Schurs.

**2.1. Words.** Throughout, we denote the set of positive integers by  $\mathbb{Z}_+$ . Given  $n \in \mathbb{Z}_+$ , define  $[n] := \{1, \dots, n\}$ . Furthermore, set  $[0] := \emptyset$ . Let  $\mathbb{Z}_+^*$  denote the set of all words in the alphabet  $\mathbb{Z}_+$ .

Consider  $w = w_1 \dots w_n \in \mathbb{Z}_+^*$ . We call  $n$  the *length* of  $w$  and denote it by  $|w|$ . The word  $w_n \dots w_1$  is denoted by  $w^r$ . We say that  $w$  is *reverse-lattice* if every suffix of  $w$  contains at least as many  $i$ 's as  $i + 1$ 's for all  $i \in \mathbb{Z}_+$ . Let  $\mathfrak{S}_n$  be the permutation group on  $[n]$  and denote the longest element in  $\mathfrak{S}_n$  by  $w_0^{(n)}$ , or simply by  $w_0$  if  $n$  is clear from context. The permutation  $\sigma \in \mathfrak{S}_n$  acts on a sequence  $\lambda = (\lambda_1, \dots, \lambda_n)$  by  $\sigma \cdot \lambda = (\lambda_{\sigma^{-1}(1)}, \dots, \lambda_{\sigma^{-1}(n)})$ . Note that this is a right action.

**2.2. Compositions and partitions.** A finite list of nonnegative integers  $\alpha = (\alpha_1, \dots, \alpha_\ell)$  is called a *weak composition*. If  $\alpha_i > 0$  for all  $1 \leq i \leq \ell$ , then  $\alpha$  is called a *composition*. If, in addition, we have  $\alpha_1 \geq \dots \geq \alpha_\ell > 0$ , then  $\alpha$  is called a *partition*. Given  $\alpha = (\alpha_1, \dots, \alpha_\ell)$  we call the  $\alpha_i$  the *parts* of  $\alpha$  and the sum of the  $\alpha_i$ , denoted by  $|\alpha|$ , is called the *size* of  $\alpha$ . We denote the number of parts of  $\alpha$  by  $\ell(\alpha)$  and call it the *length* of  $\alpha$ . The unique composition of length and size zero is denoted by  $\emptyset$ . We denote the composition  $(\alpha_\ell, \dots, \alpha_1)$  by  $\alpha^r$ . Also, denote the transpose of a partition  $\lambda$  by  $\lambda^t$ . The reader is warned that on occasion we will suppress commas and parentheses when writing our compositions.

The *composition diagram* of  $\alpha = (\alpha_1, \dots, \alpha_\ell)$  is the left-justified array of boxes with  $\alpha_i$  boxes in row  $i$  from the bottom. If  $\alpha$  is a partition, then the composition diagram of  $\alpha$  coincides with the Young diagram of  $\alpha$  in French notation. See Figure 1 for the composition diagram of  $(2, 1, 3)$ .

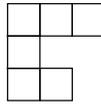


FIGURE 1. The composition diagram of  $(2, 1, 3)$ .

**2.3. Young tableaux.** A *Young tableau*  $T$  (henceforth YT) of *skew shape*  $\lambda/\mu$  is a filling of the boxes of  $\lambda/\mu$  with positive integers so that entries along the rows increase weakly read from left to right and entries along the columns increase strictly read from bottom to top. If the entries in  $T$  are all distinct and belong to  $[[\lambda/\mu]]$ , then we call  $T$  a *standard Young tableau* (henceforth SYT.) We denote the set of YTs of shape  $\lambda/\mu$  by  $\text{YT}(\lambda/\mu)$ .

If  $T$  is a YT with maximal entry  $m$ , then the *content* of  $T$ , denoted by  $\text{cont}(T)$ , is the weak composition  $(\alpha_1, \dots, \alpha_m)$  where  $\alpha_i$  for  $1 \leq i \leq m$  counts the instances of  $i$  in  $T$ . The *standardization* of  $T$ , denoted by  $\text{stan}(T)$ , is obtained by replacing the  $\alpha_i$  entries in  $T$  equal to  $i$  by the integers  $1 + \sum_{j=1}^{i-1} \alpha_j$  through  $\sum_{j=1}^i \alpha_j$  from left to right.

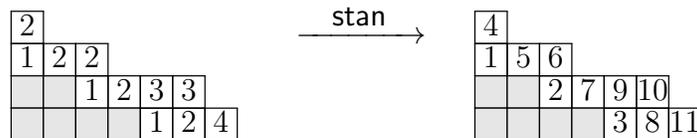


FIGURE 2. A Young tableau and its standardization.

**2.4. Composition tableaux.** To define composition tableaux, we need an analogue of Young's lattice. The *Young composition poset*  $\mathcal{L}_c$  is the poset on compositions where the partial order  $<_c$  is obtained by taking the transitive closure of the cover relation  $<_c$  defined next. Let  $\beta = (\beta_1, \dots, \beta_m)$ . Then  $\beta <_c \alpha$  if exactly one of the following conditions holds.

- $\alpha = (\beta_1, \dots, \beta_m, 1)$ .
- $\alpha = (\beta_1, \dots, \beta_k + 1, \dots, \beta_m)$  for some  $k$  where  $\beta_k \neq \beta_i$  for all  $i > k$ .

The reader may check that, for instance, the compositions covering  $(2, 1, 3, 2)$  in  $\mathcal{L}_c$  are  $(2, 1, 3, 2, 1)$ ,  $(2, 2, 3, 2)$ ,  $(2, 1, 3, 3)$  and  $(2, 1, 4, 2)$ .

**Remark 2.1.** The definition of  $\prec_c$  implies that  $\beta \prec_c \alpha$  if and only if for every  $\beta_i \geq \beta_j$  where  $i > j$  we have  $\alpha_i \geq \alpha_j$ .

If  $\beta \prec_c \alpha$  and  $\beta$  is drawn in the bottom left corner of  $\alpha$ , then the *skew composition shape*  $\alpha // \beta$  is defined to be the array of boxes that belong to  $\alpha$  but not to  $\beta$ . We refer to  $\alpha$  and  $\beta$  as the *outer shape* and *inner shape* respectively. If the inner shape is  $\emptyset$ , then instead of writing  $\alpha // \emptyset$  we simply write  $\alpha$ . The *size* of  $\alpha // \beta$ , denoted by  $|\alpha // \beta|$ , is  $|\alpha| - |\beta|$ .

A *composition tableau* (abbreviated to CT)  $\tau$  of *shape*  $\alpha // \beta$  is a filling  $\tau : \alpha // \beta \rightarrow \mathbb{Z}_+$  that satisfies the following conditions.

- (1) The entries in each row increase weakly from left to right.
- (2) The entries in the leftmost column increase strictly from bottom to top.
- (3) For any configuration in  $\tau$  of the type in Figure 3, if  $a \leq c$  then  $b < c$ .

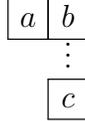


FIGURE 3. A triple configuration.

A composition tableau is *standard* if  $\tau$  is a bijection between  $\alpha // \beta$  and  $[\alpha // \beta]$ . We denote the set of CTs (respectively SCTs) of shape  $\alpha // \beta$  by  $\text{CT}(\alpha // \beta)$  (respectively  $\text{SCT}(\alpha // \beta)$ ). Figure 4 depicts a tableau in  $\text{CT}((3, 6, 1, 7) // (2, 4))$ , where the shaded boxes belong to the inner shape.

**2.5. Mason's  $\rho$  map.** Next we discuss a crucial map that bridges the combinatorics of CTs and YTs. Let  $\text{CT}(- // \beta)$  denote the set of all CTs with inner shape  $\beta$  and  $\text{YT}(- / \tilde{\beta})$  denote the set of all YTs with inner shape  $\text{sort}(\beta)$ . Then the map  $\rho_\beta : \text{CT}(- // \beta) \rightarrow \text{YT}(- / \tilde{\text{sort}}(\beta))$ , which generalizes the map for semistandard skyline fillings [27] and is introduced in [24, Chapter 4], is defined as follows. Given  $\tau \in \text{CT}(- // \beta)$ , obtain  $\rho_\beta(\tau)$  by writing the entries

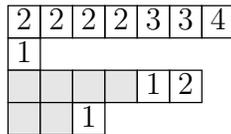


FIGURE 4. A CT of shape  $(3, 6, 1, 7) // (2, 4)$ .

in each column in increasing order from bottom to top and bottom-justifying these new columns on the inner shape  $\text{sort}(\beta)$ , which might be empty.

The inverse map  $\rho_\beta^{-1} : \text{YT}(-/\text{sort}(\beta)) \rightarrow \text{CT}(-//\beta)$  is also straightforward to define. Given  $T \in \text{YT}(-/\text{sort}(\beta))$ ,

- (1) take the set of  $i$  entries in the leftmost column of  $T$  and write them in increasing order in rows  $\ell(\beta) + 1, 2, \dots, \ell(\beta) + i$  above the inner shape  $\beta$  in the first column to form the leftmost column of  $\tau$ ,
- (2) take the set of entries in column 2 in increasing order and place them in the row with the largest index so that either
  - the box to the immediate left of the number being placed is filled and the row entries weakly increase when read from left to right, or
  - the box to the immediate left of the number being placed belongs to the inner shape,
- (3) repeat the previous step with the set of entries in column  $k$  for  $k = 3, \dots, m$  where  $m$  is the largest part of  $\text{sort}(\beta)$ .

In the case  $\beta = \emptyset$ , the map  $\rho_\beta$  is Mason's shift map [27] (or  $\rho$  map), and we set  $\rho := \rho_\emptyset$ . One may verify that the YT on the left in Figure 2 maps to the CT in Figure 6 under  $\rho_{(2,4)}^{-1}$ .

**2.6. Reading words, jdt-equivalence and rectification.** Given a word  $w = w_1 \dots w_n$ , the Robinson-Schensted correspondence (via row insertion or column insertion) associates an ordered pair  $(P(w), Q(w))$  of YTs of the same shape. We call  $P(w)$  and  $Q(w)$  the *insertion tableau* and *recording tableau* respectively. We call two words  $w_1$  and  $w_2$  *Knuth-equivalent* if  $P(w_1) = P(w_2)$ . Intimately related to the notion of Knuth-equivalence is the notion of jdt-equivalence. For this we need to introduce reading words.

**Definition 2.2** (Reading word). *The reading word of  $T \in \text{YT}(\lambda/\mu)$  (respectively  $\tau \in \text{CT}(\alpha//\beta)$ ), denoted by  $\text{rw}(T)$  (respectively  $\text{rw}(\tau)$ ), is obtained by reading the entries of  $T$  (respectively  $\tau$ ) in every column in decreasing order, starting from the leftmost column and going to the right.*

We declare YTs  $T_1$  and  $T_2$  to be *jdt-equivalent* if their reading words are Knuth-equivalent, that is,  $P(\text{rw}(T_1)) = P(\text{rw}(T_2))$ . In view of Mason's map  $\rho_\beta$ , all combinatorial notions discussed in the context of Young tableaux are inherited by composition tableaux. We focus on the notion of rectification given its importance in our context.

**Definition 2.3** (Rectification). *Given  $T \in \text{YT}(\lambda/\mu)$ , its rectification  $\text{rect}(T)$  is the unique YT of straight shape that is jdt-equivalent to  $T$ . This given, we can now define rectification for CTs. Given  $\tau \in \text{CT}(\alpha//\beta)$ , its rectification  $\text{rect}(\tau)$ , is defined to be  $\rho^{-1}(\text{rect}(\rho_\beta(\tau)))$ .*

We make note of one important consequence: Let  $\mu$  be a partition, and  $\beta^{(1)}, \beta^{(2)}$  be compositions such that  $\text{sort}(\beta^{(i)}) = \mu$  for  $i = 1, 2$ . Given  $T \in \text{YT}(\lambda/\mu)$ , let  $\tau_i = \rho_{\beta^{(i)}}^{-1}(T)$  for  $i = 1, 2$ . As  $\text{rw}(\tau_1) = \text{rw}(\tau_2)$ , we infer that  $\text{rect}(\tau_1) = \text{rect}(\tau_2)$ .

**2.7. Classical LR coefficients.** Recall that the classical LR rule provides a combinatorial way to compute the structure coefficients  $c_{\nu\mu}^\lambda$  in

$$s_\nu s_\mu = \sum_{\lambda \vdash |\mu| + |\nu|} c_{\nu\mu}^\lambda s_\lambda.$$

The LR rule was stated by Littlewood-Richardson [21] in 1934, and the first proofs were obtained by Thomas [40] and Schützenberger [35] four decades later.

**Definition 2.4.** A tableau  $T \in \text{YT}(\lambda/\mu)$  is a **Littlewood-Richardson tableau** if  $\text{cont}(T)$  is a partition and  $\text{rw}(T)$  is reverse lattice. The set of LR tableaux of shape  $\lambda/\mu$  and content  $\nu$  is denoted by  $\text{LRT}(\lambda, \mu, \nu)$ .

It transpires that the LR coefficient  $c_{\nu\mu}^\lambda$  equals  $|\text{LRT}(\lambda, \mu, \nu)|$ . Figure 5 depicts the LR tableaux that contribute to  $c_{431,644}^{76432}$ . For various combinatorial interpretations of LR coefficients, the reader is referred to [11, 20]. For a Hopf-algebraic perspective, see [17], and for a beautiful unifying polytopal perspective, see [29].

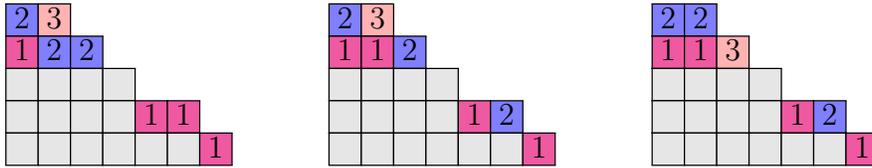


FIGURE 5. The three LR tableaux contributing to  $c_{431,644}^{76432}$ .

**2.8. Noncommutative LR coefficients.** We refer the reader to [13, 14, 24] for background on noncommutative symmetric functions and quasisymmetric functions. To describe the noncommutative LR rule, we need noncommutative analogues of Schur functions or, equivalently, quasisymmetric analogues of skew Schur functions. Following [24, Proposition 5.2.6], we define the *skew quasisymmetric Schur function* indexed by  $\alpha//\beta$  to be

$$(2.1) \quad \mathcal{S}_{\alpha//\beta} := \sum_{\tau \in \text{CT}(\alpha//\beta)} \mathbf{x}^{\text{cont}(\tau)}.$$

Here  $\mathbf{x}^{\text{cont}(\tau)} := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$  where  $m$  is the largest entry in  $\tau$  and  $(\alpha_1, \dots, \alpha_m) = \text{cont}(\tau)$ . If  $\beta = \emptyset$  in (2.1), instead of writing  $\mathcal{S}_{\alpha//\emptyset}$ , we write  $\mathcal{S}_\alpha$ . Additionally, we set  $\mathcal{S}_\emptyset = 1$ .

The noncommutative Schur functions are defined indirectly [24, Definition 5.6.1] as elements of the basis in  $\text{NSym}$  dual to the basis of quasisymmetric Schur functions in  $\text{QSym}$ . We now proceed to describe the LR rule for noncommutative Schur functions, equivalent to [24, Theorem 5.6.2].

Given a composition  $\alpha = (\alpha_1, \dots, \alpha_k)$ , the *canonical composition tableau*  $\tau_\alpha$  is constructed by filling the boxes in the  $i$ -th row of the composition diagram of  $\alpha$  with consecutive positive

integers from  $1 + \sum_{j=1}^{i-1} \alpha_j$  to  $\sum_{j=1}^i \alpha_j$  from left to right, for  $1 \leq i \leq k$ . Figure 6 shows the canonical CT of shape  $(4, 2, 3, 1)$  on the right.

10			
7	8	9	
5	6		
1	2	3	4

FIGURE 6. The canonical CT of shape  $(4, 2, 3, 1)$ .

**Theorem 2.5.** *Let  $\alpha, \beta$  be compositions. Then*

$$\mathbf{s}_\alpha \mathbf{s}_\beta = \sum_{\gamma \models |\alpha| + |\beta|} C_{\alpha\beta}^\gamma \mathbf{s}_\gamma,$$

where  $C_{\alpha\beta}^\gamma$  is the number of SCTs of shape  $\gamma // \beta$  that rectify to  $\tau_\alpha$ .

For the proof of the theorem in its original context, see [6, Theorem 3.5]. As mentioned in the introduction, one motivation for this article is to develop a polytopal perspective on noncommutative LR coefficients. Toward this end, relating them explicitly with LR tableaux is a worthwhile endeavor as the latter's description allows for a straightforward translation to linear inequalities [8, 16, 29].

**2.9. Crystal reflection operators and LR tableaux.** For in-depth exposition on crystal bases and their relevance in algebraic combinatorics and representation theory, we refer the reader to [9]. We proceed to describe crystal reflection operators.

Given a positive integer  $i$ , we define the *crystal reflection operator*  $s_i$  acting on the set of Young tableaux as follows.

- (1) Let  $T \in \text{YT}(\lambda/\mu)$ , and let  $w = w_1 \dots w_n := \text{rw}(T)$ . Scan  $w$  from left to right and pair each  $i + 1$  with the closest unpaired  $i$  that follows.
- (2) If no further pairing is possible, then change all unpaired  $i$ 's to  $i + 1$ 's or vice versa depending on whether the number of  $i$ 's is greater than the number of  $i + 1$ 's or not. Say the new word obtained via this procedure is  $w'$ .
- (3) Define  $s_i(T)$  to be the unique YT of shape  $\lambda/\mu$  such that  $\text{rw}(s_i(T)) = w'$ .

Lascoux-Schützenberger [18] (see also [32, Proposition 9] and [19, Section 3]) proved that the operators  $s_i$  give a well-defined action of the (infinite) symmetric group on  $\text{YT}(\lambda/\mu)$  by verifying the *Coxeter-Moore relations*. In particular, we obtain  $\sigma(T)$  by computing  $s_{i_1} \dots s_{i_k}(T)$  for any reduced word  $s_{i_1} \dots s_{i_k}$  for  $\sigma$ .

For  $\sigma \in \mathfrak{S}_{\ell(\nu)}$ , define

$$(2.2) \quad \text{LRT}^\sigma(\lambda, \mu, \nu) := \{\sigma(T) \mid T \in \text{LRT}(\lambda, \mu, \nu)\}.$$

Since crystal reflection operators define an  $\mathfrak{S}_{|\ell(\nu)|}$ -action, we have  $|\text{LRT}^\sigma(\lambda, \mu, \nu)|$  equals  $c_{\nu\mu}^\lambda$ , for all permutations  $\sigma \in \mathfrak{S}_{\ell(\nu)}$ .

Figure 7 shows all tableaux in  $\text{LRT}^\sigma(\lambda, \mu, \nu)$  where  $\lambda = (7, 6, 4, 3, 2)$ ,  $\mu = (6, 4, 4)$ ,  $\nu = (4, 3, 1)$  and  $\sigma = s_1 s_2$ . The tableaux in  $\text{LRT}(\lambda, \mu, \nu)$  are shown in Figure 5. Note that all tableaux in Figure 7 have content  $(1, 4, 3) = \sigma \cdot (4, 3, 1)$ .

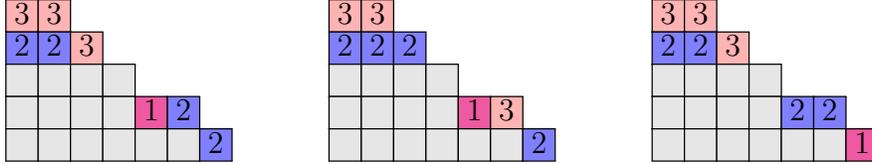


FIGURE 7. Crystal reflection corresponding to  $s_1 s_2$  on the tableaux in Figure 5.

The crucial insight behind our work is that even though  $\text{LRT}^\sigma(\lambda, \mu, \nu)$  sheds no further light on classical LR coefficients, it ‘knows’ about the noncommutative LR coefficients. To motivate our upcoming results and to establish a connection with Theorem 2.5, we invite the reader to check that for all tableaux  $T$  in Figure 7, we have that  $\rho^{-1}(\text{rect}(T))$  is the distinguished CT in Figure 8 with the defining property that it is the unique CT with shape and content both equaling  $(1, 4, 3)$ . Equivalently, we could say that  $\rho^{-1}(\text{rect}(\text{stan}(T)))$  is the canonical CT of shape  $(1, 4, 3)$ .

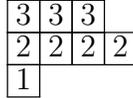


FIGURE 8. The rectification of all tableaux in Figure 7.

In Section 4, we establish this property in general by exploiting the interplay between frank words and jeu-de-taquin. This will allow us to establish the veracity of our main theorem.

### 3. MAIN RESULT AND RELATED DISCUSSION

Our main result provides a combinatorial description for  $C_{\alpha\beta}^\gamma$  using crystal reflection operators. We state it next and give a proof assuming results from Sections 4 and 5.

**Theorem 3.1.** *Let  $\alpha$ ,  $\beta$  and  $\gamma$  be compositions such that  $|\beta| + |\alpha| = |\gamma|$ . Let  $(\lambda, \mu, \nu)$  be the triple  $(\text{sort}(\gamma), \text{sort}(\beta), \text{sort}(\alpha))$ . If  $\sigma \in \mathfrak{S}_{\ell(\nu)}$  is a permutation such that  $\alpha = \sigma \cdot \nu$ , then*

$$C_{\alpha\beta}^\gamma = |\{T \in \text{LRT}^\sigma(\lambda, \mu, \nu) \mid \text{shape}(\rho_\beta^{-1}(T)) = \gamma // \beta\}|$$

*Proof.* Let  $(\lambda, \mu, \nu)$  be as in the statement of the theorem. Recall from (1.3) in the introduction that

$$(3.1) \quad c_{\nu\mu}^\lambda = \sum_{\text{sort}(\delta)=\lambda} C_{\alpha\beta}^\delta,$$

where we work with the convention that  $C_{\alpha\beta}^\delta = 0$  if  $\delta$  does not lie above  $\beta$  in  $\mathcal{L}_c$ . Given  $\gamma$  such that  $\text{sort}(\gamma) = \lambda$ , define

$$(3.2) \quad X_\gamma := \{T \in \text{LRT}^\sigma(\lambda, \mu, \nu) \mid \text{shape}(\rho_\beta^{-1}(\text{stan}(T))) = \gamma\}.$$

By Proposition 5.3, we have that  $|X_\gamma| \leq C_{\alpha\beta}^\gamma$ . On the other hand, from (3.1) and the fact that  $\text{shape}(\rho_\beta^{-1}(\text{stan}(T))) = \text{shape}(\rho_\beta^{-1}(T))$  for all YTs  $T$  with inner shape  $\mu$ , we infer that

$$(3.3) \quad |\text{LRT}^\sigma(\lambda, \mu, \nu)| = \sum_{\text{sort}(\delta)=\lambda} |X_\delta|.$$

Thus, we must have  $|X_\gamma| = C_{\alpha\beta}^\gamma$ . □

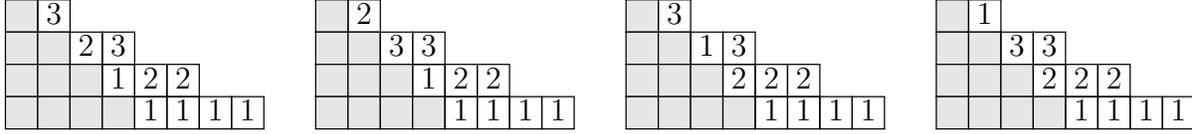
**3.1. Comparison with known results.** It is natural to inquire whether our Theorem 3.1 affords further insight into computing noncommutative LR coefficients when compared to the already known description of Bessenrodt-Luoto-van Willigenburg [6]. To this end, note that [24, Theorem 5.6.2] offers little algorithmic insight in terms of judiciously picking SCTs of shape  $\gamma // \beta$  that rectify to  $\tau_\alpha$ , other than an exhaustive search. Additionally, given the involved bumping algorithm for jeu-de-taquin slides on composition tableaux [38], the efficient approach to rectification is still to perform Mason’s insertion algorithm.

In contrast, Theorem 3.1 emphasizes the curious aspect that computing noncommutative LR coefficients is best done by passing through the commutative realm. Indeed, although computing classical LR coefficients is known to be  $\#P$ -hard, we can appeal to one of many descriptions that they possess to compute them in a deterministic manner; see [8, 16, 22, 41]. Once the LR tableaux in  $\text{LRT}(\lambda, \mu, \nu)$  are computed, it remains to apply an appropriate sequence of crystal reflections (determined by  $\alpha$ ) followed by an appropriate  $\rho_\beta$  to determine *all* noncommutative LR coefficients  $C_{\alpha\beta}^\gamma$ .

On a related note, Bessenrodt-Tewari-van Willigenburg [7], in providing a classification of symmetric skew quasisymmetric Schur functions, established Yamanouchi-type rules—called *left* and *right* LR rules therein—for the noncommutative LR coefficients  $C_{\alpha\beta}^\gamma$  in the special cases where  $\alpha$  is either a partition or a reverse partition, although the proof therein was not uniform. It transpires that these two rules are in fact two extremes of our Theorem 3.1. Indeed by picking  $\sigma$  to either be the identity or the longest word in  $\mathfrak{S}_{\ell(\nu)}$ , we obtain the rules of [7]; see discussion at the end of Section 5 for details.

**3.2. Illustrating our main result.** We discuss an example next to illustrate Theorem 3.1. Let  $\lambda = 8642$ ,  $\mu = 4321$  and  $\nu = 532$ . Figure 9 shows tableaux in  $\text{LRT}(\lambda, \mu, \nu)$ , implying in particular that  $c_{532,4321}^{8642} = 4$ .

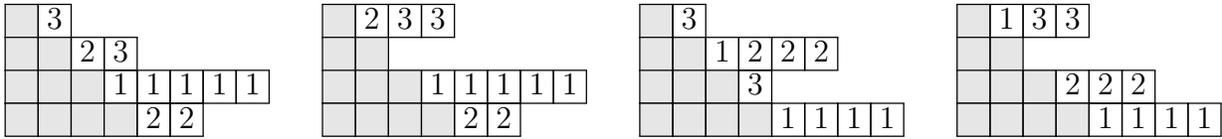
**Example 3.2.** Suppose we pick  $\sigma \in \mathfrak{S}_3$  to be the identity permutation and  $\beta = \mu$ . Thus we do not need to apply crystal reflections. By applying  $\rho_{4321}^{-1}$  to the LR tableaux in Figure 9

FIGURE 9. The tableaux in  $LRT(8642, 4321, 532)$ .

we obtain the CTs in Figure 10. By grouping these composition tableaux according to their outer shapes, we infer that

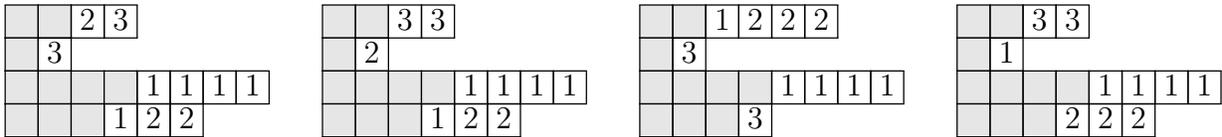
$$C_{532,4321}^{6842} = C_{532,4321}^{6824} = C_{532,4321}^{8462} = C_{532,4321}^{8624} = 1.$$

In particular we have  $C_{532,4321}^{\gamma} = 0$  for all rearrangements of  $\lambda$  not in  $\{6842, 6824, 8462, 8624\}$ .

FIGURE 10. CTs obtained by applying  $\rho_{4321}^{-1}$  to tableaux in Figure 9.

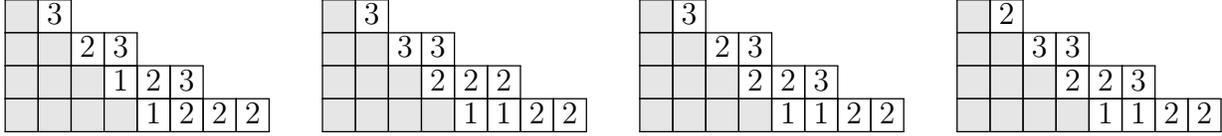
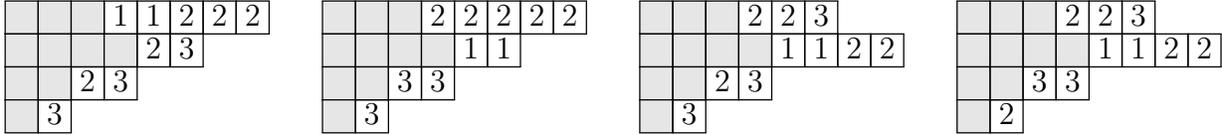
**Example 3.3.** Sticking with the same  $\sigma$  as before, we now pick  $\beta = 3412$ . The CTs resulting from applying  $\rho_{3412}^{-1}$  are shown in Figure 11. Grouping these tableaux according to their outer shape reveals a different decomposition of  $c_{532,4321}^{8642}$ . More precisely, we obtain

$$C_{532,3412}^{6824} = 3, \text{ and } C_{532,3412}^{4826} = 1.$$

FIGURE 11. CTs obtained by applying  $\rho_{3412}^{-1}$  to tableaux in Figure 9.

**Example 3.4.** Suppose we now pick  $\sigma = s_1 s_2$ . Then  $\alpha = \sigma \cdot \nu = (2, 5, 3)$  and the four tableaux in  $LRT^\sigma(\lambda, \mu, \nu)$  are depicted in Figure 12. We pick  $\beta = 1243$ . Applying  $\rho_{1243}^{-1}$  to these tableaux gives us the CTs in Figure 13. This time we obtain

$$C_{253,1243}^{2468} = 2, \text{ and } C_{253,1243}^{2486} = 2.$$

FIGURE 12. The action of  $s_1s_2$  on tableaux in Figure 9.FIGURE 13. CTs obtained by applying  $\rho_{1243}^{-1}$  to tableaux in Figure 12.

#### 4. FRANK WORDS AND LR TABLEAUX

In order to prove Theorem 3.1, we need to understand the rectification of tableaux in  $\text{LRT}^\sigma(\lambda, \mu, \nu)$  followed by an application of the generalized  $\rho$  map. To this end, it helps to explore the relation between growth words of these tableaux and frank words.

##### 4.1. Growth words and frank words.

**Definition 4.1** (Growth word). *Given an SYT  $T$ , its **growth word**  $\text{gw}(T)$  is obtained by reading entries from largest to smallest and recording the columns they belong to. We extend the definition of growth word to all YTs by setting  $\text{gw}(T) := \text{gw}(\text{stan}(T))$ .*

The reader may verify that for both tableaux in Figure 2, the growth word is 76564321531.

Frank words were introduced by Lascoux-Schützenberger [19] in their investigation of key polynomials. Subsequently, Reiner and Shimozono [30] studied the combinatorics of frank words in depth in the context of a flagged LR rule, and we follow their exposition as far as notions in this section are concerned.

We call  $w = w_1 \cdots w_n \in \mathbb{Z}_+^*$  a **column word** if  $w_1 > \cdots > w_n$ . Given a nonempty word  $w \in \mathbb{Z}_+^*$ , consider its factorization  $w^{(1)}w^{(2)} \cdots w^{(m)}$  where each  $w^{(i)}$  is a maximal column word (necessarily nonempty). We call  $w$  an  **$m$ -column word**. The **column form** of  $w$  is the composition  $\text{colform}(w) := (|w^{(1)}|, \dots, |w^{(m)}|)$ .

**Definition 4.2** (Frank word). *We say that  $w$  is **frank** if  $\text{P}(w)$  is of shape  $\lambda^t$  where  $\lambda = \text{sort}(\text{colform}(w))$ . Given a composition  $\alpha$ , we define  $\text{Frank}(\alpha)$  to be the set of frank words  $w$  satisfying  $\text{colform}(w) = \alpha$ .*

A remarkable feature of frank words is that one knows the shape of the P-tableau without performing jeu-de-taquin. As an example of a frank word, consider  $w = \boxed{432} \boxed{32} \boxed{6531}$ . It is a 3-column word with  $\text{colform}(w) = (3, 2, 4)$ . For clarity, we put frames around maximal columns words. Figure 14 depicts  $\text{P}(w)$ . Note that the shape underlying it is  $(4, 3, 2)^t$ . Therefore,  $w \in \text{Frank}((3, 2, 4))$ .

4		
3	6	
2	3	5
1	2	3

FIGURE 14. The insertion tableau corresponding to  $w = \boxed{432} \boxed{32} \boxed{6531}$ .

4.2. **Outline of proof.** Given the technical nature of our argument, we give a brief overview of our strategy toward establishing Theorem 3.1.

- (1) Given a skew shape  $\lambda/\mu$  and a composition  $\alpha \vDash |\lambda| - |\mu|$ , we consider the distinguished subset of  $\text{Frank}(\alpha)$  comprising  $\lambda/\mu$ -compatible frank words.
- (2) The main result of this section, Proposition 4.6, identifies these words as precisely the growth words of tableaux in  $\text{LRT}^\sigma(\lambda, \mu, \nu)$  where  $\nu$  satisfies  $\text{sort}(\alpha) = \nu$ . To arrive at this result, we study a symmetric group action on frank words and establish that it really is the crystal reflection action in disguise. This is the content of Subsections 4.4 and 4.5.
- (3) Lemma 5.1 then states that the rectification of a Young tableau may be described in terms of its growth word using Schützenberger’s evacuation.
- (4) Finally, Proposition 5.3 establishes that applying  $\rho^{-1}$  to  $\text{rect}(\text{stan}(T))$  for  $T$  an element in  $\text{LRT}^\sigma(\lambda, \mu, \nu)$  does indeed give the canonical tableau of shape  $\sigma \cdot \nu$ . From this, Theorem 3.1 follows.

#### 4.3. $\lambda/\mu$ -compatible frank words and LR tableaux.

**Definition 4.3.** *Given partitions  $\lambda, \mu$  such that  $\mu \subseteq \lambda$  and a composition  $\alpha$ , we say that  $w \in \text{Frank}(\alpha)$  is  $\lambda/\mu$ -compatible if for every suffix  $w'$  of  $w$ , we have that  $\text{cont}(w') + \mu^t$  is a partition and that  $\text{cont}(w) + \mu^t = \lambda^t$ . Set*

$$\text{Frank}(\lambda, \mu, \alpha) := \{w \in \text{Frank}(\alpha) \mid w \text{ is } \lambda/\mu\text{-compatible}\}.$$

Definition 4.3 may be interpreted as saying that  $w \in \text{Frank}(\alpha)$  is  $\lambda/\mu$ -compatible if it is the growth word of a Young tableau of shape  $\lambda/\mu$ . Observe that we must have  $\alpha \vDash |\lambda| - |\mu|$ .

As an example, note that  $w = \boxed{621} \boxed{76432} \in \text{Frank}(\alpha)$  is  $\lambda/\mu$ -compatible for  $\lambda = 76422$ ,  $\mu = 5521$ , and  $\alpha = 35$ . The reader can easily verify that  $w$  is the growth word of the tableau in Figure 15. Note further that this tableau is an LR tableau.

There is an intimate link between LR tableaux and compatible words. To understand this link we go via an intermediate tableau object.

Given  $w \in \text{Frank}(\lambda, \mu, \alpha)$ , let  $w^{(1)} \cdots w^{(m)}$  be its maximal column word factorization where  $m := \ell(\alpha)$ . Construct a Young tableau  $\phi(w)$  of shape  $\lambda/\mu$  and content  $\alpha^r$  as follows: Let  $\lambda^{(0)} := \lambda$  and inductively define  $\lambda^{(i)}$  for  $1 \leq i \leq m$  to be such that  $\lambda^{(i-1)}/\lambda^{(i)}$  is a horizontal strip with boxes in columns given by letters appearing in  $w^{(i)}$ . Subsequently, fill the boxes

2	2						
	1						
		1	1				
					2		
					1	1	

FIGURE 15. The tableau  $\phi(w)$  corresponding to  $\boxed{621} \boxed{76432}$ .

of the horizontal strips  $\lambda^{(i-1)}/\lambda^{(i)}$  with  $m+1-i$  to obtain  $\phi(w)$ . Note that  $\lambda^{(m)}$  is  $\mu$  and that  $\phi(w)$  does indeed belong to  $\text{YT}(\lambda/\mu)$ . Figure 15 gives an example.

Define

$$(4.1) \quad \text{FrankTab}(\lambda, \mu, \alpha) := \{\phi(w) \mid w \in \text{Frank}(\lambda, \mu, \alpha)\}.$$

Our next lemma establishes that  $\text{LRT}(\lambda, \mu, \nu) = \text{FrankTab}(\lambda, \mu, \nu^r)$ .

**Lemma 4.4.** *Let  $\lambda, \mu$  and  $\nu$  be partitions such that  $\mu \subseteq \lambda$  and  $|\nu| = |\lambda/\mu|$ . We have*

$$\text{LRT}(\lambda, \mu, \nu) = \text{FrankTab}(\lambda, \mu, \nu^r).$$

*Proof.* Let  $w \in \text{Frank}(\lambda, \mu, \nu^r)$ . As  $\text{colform}(w) = \nu^r$  and  $w$  is frank, we know that  $w$  is the reading word of a tableau of skew shape  $\text{rotate}(\nu^t)$ , which is obtained from the shape  $\nu$  by a  $180^\circ$  rotation. It follows that  $\phi(w) \in \text{LRT}(\lambda, \mu, \nu)$ . This establishes  $\text{FrankTab}(\lambda, \mu, \nu^r) \subseteq \text{LRT}(\lambda, \mu, \nu)$ .

The opposite inclusion follows since the reading word of an LR tableau is reverse-lattice. In particular, the growth word of any  $T \in \text{LRT}(\lambda, \mu, \nu)$  is the reading word of a YT of skew shape  $\text{rotate}(\nu^t)$ .  $\square$

Figure 16 depicts growth words of LR tableaux in Figure 5 as reading words of YTs of skew shape  $\text{rotate}(\nu^t)$ . Here  $\lambda = 76432$ ,  $\mu = 644$  and  $\nu = 431$ .

2	3	7			
	2	6			
	1	5			
		1			

2	6	7			
	3	5			
	1	2			
		1			

3	6	7			
	2	5			
	1	2			
		1			

FIGURE 16. Growth words of LR tableaux are frank words.

Now that we have established elements of  $\text{FrankTab}(\lambda, \mu, \nu^r)$  as LR tableaux in the usual sense, it remains to understand the case where  $\nu^r$  is replaced by an arbitrary composition.

**4.4. A symmetric group action on frank words.** We proceed to describe a symmetric group action on frank words that we subsequently connect to the symmetric group action on Young tableaux via crystal reflections described earlier. This new action is best understood by focusing on 2-column frank words, very much in the manner that crystal reflections are understood by acting on consecutive letters.

Let  $A$  denote the set of 2-column frank words. Consider  $w \in A$  and let  $\text{colform}(w) = (\beta_1, \beta_2)$ . Thus,  $w$  may be identified as the reading word of a tableau  $T$  of shape  $(\beta_1, \beta_2)^t$  if  $\beta_1 \geq \beta_2$  or of a tableau  $T$  of shape  $(\beta_2, \beta_2)^t / (\beta_2 - \beta_1)^t$  if  $\beta_1 < \beta_2$ ; see [30, Appendix 2].

If the former holds, define  $\iota(w)$  to be the reading word of the unique tableau  $T'$  of shape  $(\beta_1, \beta_1)^t / (\beta_1 - \beta_2)^t$  that is jdt-equivalent to  $T$  (obtained by performing jeu-de-taquin slides within the rectangle  $(\beta_1, \beta_1)^t$ ). If the latter holds, define  $\iota(w)$  to be the reading word of the unique tableau  $T'$  of shape  $(\beta_2, \beta_1)^t$  that is jdt-equivalent to  $T$ .

Clearly,  $\iota$  is an involution on  $A$ . Equally importantly,  $w$  and  $\iota(w)$  are Knuth-equivalent. For instance, jdt-equivalence of tableaux in Figure 17 implies that

$$\iota(\boxed{76421} \boxed{632}) = \boxed{621} \boxed{76432}.$$

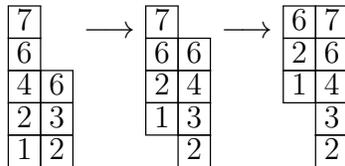


FIGURE 17. Establishing  $\iota(\boxed{76421} \boxed{632}) = \boxed{621} \boxed{76432}$  by jeu-de-taquin.

We employ the involution  $\iota$  to construct the desired symmetric group action. Let  $\lambda$  be a partition and let  $m := \ell(\lambda)$ . Define

$$\mathcal{F}_\lambda := \coprod_{\text{sort}(\beta)=\lambda} \text{Frank}(\beta).$$

Following [19], define an action of  $\mathfrak{S}_m$  on  $\mathcal{F}_\lambda$  by describing the action of the generator  $s_i$  for  $1 \leq i \leq m - 1$  as follows: Let  $w^{(1)} \dots w^{(m)}$  be the maximal column word factorization of  $w \in \mathcal{F}_\lambda$ . Let  $v^{(i)} v^{(i+1)} := \iota(w^{(i)} w^{(i+1)})$  and define  $v = w^{(1)} \dots w^{(i-1)} v^{(i)} v^{(i+1)} w^{(i+2)} \dots w^{(m)}$ . Observe that  $\text{sort}(\text{colform}(v)) = \lambda$ . As  $v$  and  $w$  are Knuth-equivalent, we infer that  $v \in \mathcal{F}_\lambda$ . We define  $v$  to be  $s_i(w)$ . This given, we may define  $\sigma(w)$  for any  $\sigma \in \mathfrak{S}_m$  by picking a reduced word for  $\sigma$ .

Figure 18 depicts the  $\mathfrak{S}_3$ -orbit of the frank word  $\boxed{4321} \boxed{632} \boxed{53} \in \mathcal{F}_{432}$ . We have chosen to represent the action on words via their tableau representatives. In particular, each column when read from top to bottom gives a maximal column word in the maximal column word factorization. Each  $s_i$  performs a ‘local’ jeu-de-taquin on two columns.

**4.5. Relating the two symmetric group actions.** Our next lemma connects the action of crystal reflection operators on  $\text{FrankTab}(\lambda, \mu, \alpha)$  to the symmetric group action on  $\lambda/\mu$ -compatible frank words in  $\text{Frank}(\lambda, \mu, \alpha)$ .

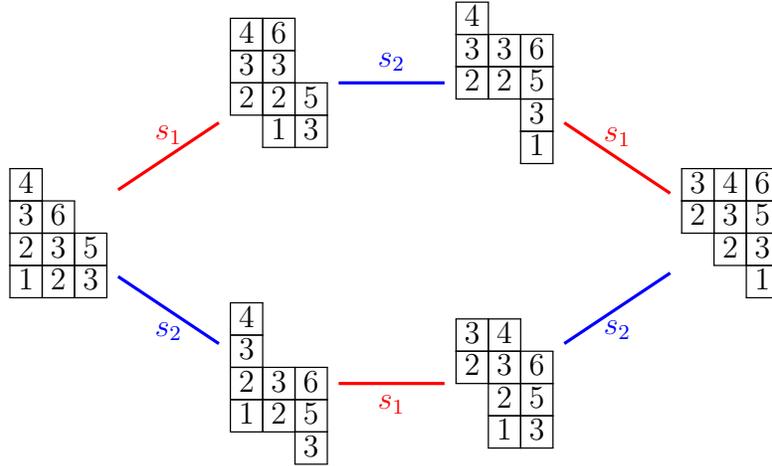


FIGURE 18. The  $\mathfrak{S}_3$ -orbit of  $\boxed{4321} \boxed{632} \boxed{53}$ .

**Lemma 4.5.** *Consider  $w \in \text{Frank}(\lambda, \mu, \alpha)$  and  $i$  satisfying  $1 \leq i \leq \ell(\alpha) - 1$ . We have that  $s_i(\phi(w)) = \phi(s_{\ell(\alpha)-i}(w))$ .*

*Proof.* Without loss of generality, we may assume that  $\alpha$  has two parts. Suppose  $\alpha = (p, q) \vDash n$ . Assume  $w = w^{(1)}w^{(2)}$  where  $|w^{(1)}| = p$  and  $|w^{(2)}| = q$ . We want to show that  $s_1(\phi(w)) = \phi(\iota(w))$ .

Assume  $p < q$ . Let  $w^{(1)} = a_1 \dots a_p$  and  $w^{(2)} = b_1 \dots b_q$ . We have  $a_1 > \dots > a_p$  and  $b_1 > \dots > b_q$ . As  $w$  is frank, it is the reading word of a Young tableau of skew shape  $\text{rotate}((q, p)^t)$ . Thus,  $a_i \leq b_i$  for  $1 \leq i \leq p$ .

Instead of computing  $\iota(w)$  by rectifying the appropriate two-columned tableau, one may perform successive Schensted column insertions of the numbers  $a_p$  down to  $a_1$  starting from the single-columned tableau with column word  $w^{(2)}$ . See Figure 19 for an example. Compare with Figure 17 which established the same fact using jeu-de-taquin.

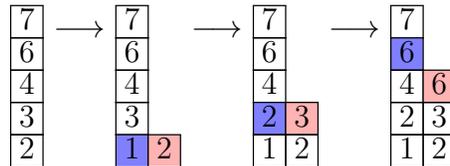


FIGURE 19. Establishing that  $\iota(\boxed{621} \boxed{76432}) = \boxed{76421} \boxed{632}$  by column-inserting 1, 2, and 6 into the single-columned tableau with reading word 76432. Blue boxes show the entries being inserted whereas red boxes contain the entries bumped.

During each intermediate step of this column insertion procedure, the number  $a_i$  being inserted into the current tableau bumps a distinct element from  $\{b_1, \dots, b_q\}$ . Furthermore this bumped entry is guaranteed to be strictly greater than the entries in the second column in the current tableau. Therefore, the insertion tableau is completely determined by the entries that get bumped. More precisely, for  $i$  from  $p$  down to 1, define the integer  $m(i)$  recursively as follows. We define  $m(p)$  to be the largest integer  $j$  such that  $a_p \leq b_j$ . Subsequently, for  $i = p - 1, \dots, 1$ , define  $m(i)$  to be the largest integer  $j$  such that  $j < m(i + 1)$  and  $a_i \leq b_j$ . Observe that in our Schensted column-insertion procedure, the entry  $a_i$  bumps  $b_{m(i)}$ . Therefore, the set of entries that get bumped is  $\{b_{m(i)} \mid 1 \leq i \leq p\}$ . For the example in Figure 19, we have  $m(3) = 5$ ,  $m(2) = 4$  and  $m(1) = 2$ . Therefore the set of entries that get bumped is  $\{b_5, b_4, b_2\} = \{2, 3, 6\}$ .

Consider  $\text{rw}(\phi(w)) = u_1 \dots u_n$ . The word  $v := v_1 \dots v_n$  obtained by recording the column to which each  $u_i$  belongs gives us the weakly increasing arrangement of letters in  $w$ . Furthermore, for  $1 \leq i \leq p$  (respectively  $1 \leq i \leq q$ ) the letter in  $v$  corresponding to the  $i$ th 2 (respectively 1) from the left in  $\text{rw}(\phi(w))$  is equal to  $a_i$  (respectively  $b_i$ ). Recall that the crystal reflection operator  $s_1$  acting on  $\phi(w)$  begins by pairing each 2 in  $\text{rw}(\phi(w))$  to the closest unpaired 1 to its right. Equivalently, in our current context, a 2 corresponding to  $a_i$  for some  $1 \leq i \leq p$  gets paired with the 1 in  $\text{rw}(\phi(w))$  corresponding to  $b_{m(i)}$ . We infer that the unpaired 1s in  $\text{rw}(\phi(w))$  correspond to those  $b_j$  that are not bumped. These are precisely the  $b_j$  that determine which 1s in  $\phi(w)$  turn into 2s in computing  $s_1(\phi(w))$ . Thus we infer that  $s_1(\phi(w)) = \phi(\iota(w))$ . This establishes the claim in the case  $p < q$ . The case  $p \geq q$  is similar and left to the reader.  $\square$

To illustrate the ideas in the preceding proof, Figure 20 depicts the action of  $s_1$  on the tableau  $\phi(w)$  from Figure 15, where  $w = \boxed{621} \boxed{76432}$ . From Figure 19, we see that the entries that do not get bumped are  $\{4, 7\}$ . Also, note that  $\text{rw}(\phi(w)) = 22111211$  where the unpaired 1s are highlighted. In terms of the tableau  $\phi(w)$ , we see that the unpaired 1s belong to columns 4 and 7. The tableau on the right in Figure 20 is easily verified to be  $\phi(\iota(w))$  as  $\iota(w) = \boxed{76421} \boxed{632}$ .

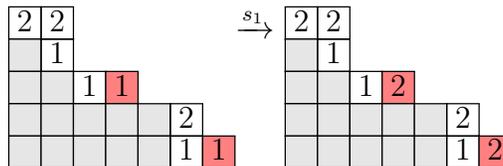


FIGURE 20. Crystal reflection on  $\phi(w)$  and its relation to the  $\iota$  involution.

We are ready to give a precise relation between  $\text{LRT}^\sigma(\lambda, \mu, \nu)$  and  $\lambda/\mu$ -compatible frank words with a certain column form that generalizes Lemma 4.4.

**Proposition 4.6.** *Let  $\lambda, \mu$  and  $\nu$  be partitions such that  $\mu \subseteq \lambda$  and  $|\nu| = |\lambda/\mu|$ . Let  $w_0$  be the longest word in  $\mathfrak{S}_{\ell(\nu)}$ . For  $\sigma \in \mathfrak{S}_{\ell(\nu)}$ , we have*

$$\text{LRT}^\sigma(\lambda, \mu, \nu) = \text{FrankTab}(\lambda, \mu, (w_0\sigma w_0) \cdot \nu^r).$$

*Proof.* Lemma 4.4 implies  $\text{LRT}(\lambda, \mu, \nu) = \text{FrankTab}(\lambda, \mu, \nu^r)$ . Consider  $T \in \text{LRT}^\sigma(\lambda, \mu, \nu)$ . We must have  $T = \sigma(T')$  for a unique  $T' \in \text{LRT}(\lambda, \mu, \nu)$ , which in turn implies that  $T = \sigma(\phi(w'))$  for a unique  $w' \in \text{Frank}(\lambda, \mu, \nu^r)$ . Lemma 4.5 implies that  $T = \phi(w_0\sigma w_0(w'))$ . As  $\text{colform}(w_0\sigma w_0(w')) = (w_0\sigma w_0) \cdot \nu^r$ , we conclude that

$$(4.2) \quad \text{LRT}^\sigma(\lambda, \mu, \nu) \subseteq \text{FrankTab}(\lambda, \mu, (w_0\sigma w_0) \cdot \nu^r).$$

A simple cardinality count implies this inclusion must be an equality.  $\square$

The upshot of Proposition 4.6 is that elements of  $\text{LRT}^\sigma(\lambda, \mu, \nu)$  are in bijection with  $\lambda/\mu$ -compatible frank words with column form  $w_0\sigma w_0 \cdot \nu^r$ . As an example, consider  $\text{LRT}^\sigma(\lambda, \mu, \nu)$  from Figure 7 where  $\lambda = (7, 6, 4, 3, 2)$ ,  $\mu = (6, 4, 4)$ ,  $\nu = (4, 3, 1)$  and  $\sigma = s_1s_2$ . Note that growth words of the tableaux therein are indeed frank words with column form  $(w_0s_1s_2w_0) \cdot (1, 3, 4) = (s_2s_1) \cdot (1, 3, 4) = (3, 4, 1)$ . Figure 21 shows these frank words as reading words of tableaux with column lengths 3, 4 and 1 read from left to right. We encourage the reader to obtain these tableaux by performing jeu-de-taquin slides to tableaux in Figure 16.

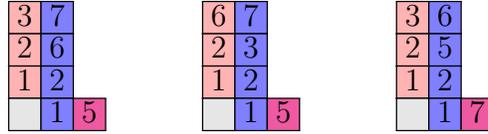


FIGURE 21. Growth words of tableaux in Figure 7 are frank words.

## 5. NONCOMMUTATIVE LR COEFFICIENTS AND FRANK WORDS

Now that we understand tableaux in  $\text{LRT}^\sigma(\lambda, \mu, \nu)$  as certain  $\lambda/\mu$ -compatible words with a prescribed column form, we are ready to establish the connection to noncommutative LR coefficients. Recall that our goal is to show that  $\text{rect}(\text{stan}(T))$  for  $T \in \text{LRT}^\sigma(\lambda, \mu, \nu)$  is equal to the canonical CT of shape  $\sigma \cdot \nu$ . We recast  $\text{rect}(\text{stan}(T))$  in terms of the growth word of  $T$ .

**Lemma 5.1.** *For any Young tableau  $T$ , we have  $\text{rect}(\text{stan}(T)) = \text{evac}(\mathbf{Q}(\text{gw}(T)))^t$ .*

*Proof.* We sketch the proof and follow the exposition in [23]. Let  $T' = \text{stan}(T)$  and suppose that  $T$  has  $n$  boxes. Consider the biword  $\begin{bmatrix} u \\ v \end{bmatrix}$  where  $u := u_1 \cdots u_n$  is the longest word in  $\mathfrak{S}_n$  and  $v := v_1 \cdots v_n$  is obtained by recording the column in  $T'$  to which  $u_i$  belongs. In other words,  $v = \text{gw}(T') = \text{gw}(T)$ . Consider biwords  $\begin{bmatrix} u' \\ v' \end{bmatrix}$  and  $\begin{bmatrix} u'' \\ v'' \end{bmatrix}$ , where  $u'$  (respectively  $v''$ )

in the weakly increasing rearrangement of  $u$  (respectively  $v$ ) and  $v'$  (respectively  $u''$ ) is the rearrangement induced by the aforementioned sorting. Then we have that

$$v' = \mathbf{gw}(T')^r \text{ and } u'' = \mathbf{rw}(T').$$

Note that  $\mathbf{rect}(\mathbf{stan}(T)) = \mathbf{rect}(T') = \mathbf{P}(\mathbf{rw}(T')) = \mathbf{P}(u'')$ . By [23, Proposition 5.3.9], this equals  $\mathbf{Q}(v')$  and by [37, Corollary A1.2.11] which relates reversal to evacuation, the claim now follows.  $\square$

For the leftmost tableau  $T$  in Figure 22, its standardization rectifies to the tableau in the middle. We have  $\mathbf{gw}(T) = \boxed{321} \boxed{7621} \boxed{5}$ , and  $\mathbf{Q}(\mathbf{gw}(T))$  is shown on the right. We invite the reader to verify that  $\mathbf{evac}(\mathbf{Q}(\mathbf{gw}(T)))$  is indeed the tableau in the middle upon transposing.

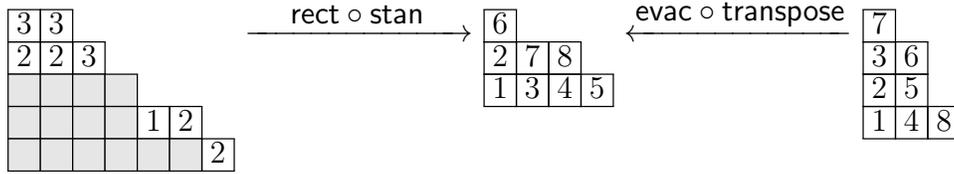


FIGURE 22. A demonstration of the claim  $\mathbf{rect}(\mathbf{stan}(T)) = \mathbf{evac}(\mathbf{Q}(\mathbf{gw}(T)))^t$ .

We now appeal to a simple defining characterization of canonical composition tableaux that will be useful. Recall that the *descent set* of an SYT  $T$  with  $n$  boxes is the set of integers  $i \in [n - 1]$  such that  $i + 1$  occupies a row strictly above that occupied by  $i$ . Via the folklore bijection between compositions of  $n$  and subsets of  $[n - 1]$  that sends  $\alpha = (\alpha_1, \dots, \alpha_\ell)$  to  $\mathbf{set}(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \dots + \alpha_{\ell-1}\}$ , one obtains the *descent composition* of  $T$  from its descent set.

**Lemma 5.2.** *Let  $\alpha$  be a composition. There is a unique Young tableau  $T$  of shape  $\mathbf{sort}(\alpha)$  and descent composition  $\alpha$ . Furthermore, this tableau satisfies  $\rho^{-1}(T) = \tau_\alpha$ .*

*Proof.* Observe that the first column of  $T$  when read from bottom to top is forced to be  $1, 1 + \alpha_1, 1 + \alpha_1 + \alpha_2, \dots, 1 + \sum_{j=1}^{\ell(\alpha)-1} \alpha_j$ . The first claim follows from this observation, and our second claim follows from applying the map  $\rho$  to  $\tau_\alpha$ . We omit the details.  $\square$

We finally arrive at the key proposition that is utilized in the proof of our central result Theorem 3.1.

**Proposition 5.3.** *If  $T \in \mathbf{LRT}^\sigma(\lambda, \mu, \nu)$ , then  $\mathbf{rect}(\mathbf{stan}(T)) = \rho(\tau_\alpha)$  where  $\alpha = \sigma \cdot \nu$ .*

*Proof.* Throughout, assume that  $\sigma \cdot \nu = \alpha$  and set  $n := |\alpha|$ . By Lemma 5.1 we have

$$(5.1) \quad \mathbf{rect}(\mathbf{stan}(T)) = \mathbf{evac}(\mathbf{Q}(\mathbf{gw}(T)))^t.$$

Since  $T \in \mathbf{LRT}^\sigma(\lambda, \mu, \nu)$ , Proposition 4.6 implies

$$(5.2) \quad \mathbf{gw}(T) \in \mathbf{Frank}(\lambda, \mu, \alpha^r).$$

Therefore, the shape underlying  $\mathbf{Q}(\mathbf{gw}(T))$  is  $\mathbf{sort}(\alpha)^t = \nu^t$ . Using the preceding fact along with (5.1), we infer that  $\mathbf{rect}(\mathbf{stan}(T))$  has shape  $\nu$ .

Note that the descent set of  $\mathbf{gw}(T)$  is  $\{n-i \mid i \in [n-1] \setminus \mathbf{set}(\alpha)\}$ , which therefore is also the descent set of  $\mathbf{Q}(\mathbf{gw}(T))$ . It follows that the descent set of  $\mathbf{evac}(\mathbf{Q}(\mathbf{gw}(T)))$  is  $[n-1] \setminus \mathbf{set}(\alpha)$ , which in turn implies that the descent set of  $\mathbf{rect}(\mathbf{stan}(T)) = \mathbf{evac}(\mathbf{Q}(\mathbf{gw}(T)))^t$  is  $\mathbf{set}(\alpha)$ . Thus we have established that  $\mathbf{rect}(\mathbf{stan}(T))$  has shape  $\mathbf{sort}(\alpha)$  and descent composition  $\alpha$ . Lemma 5.2 proves the proposition.  $\square$

This also completes the proof of our main theorem. A remarkable aspect of Proposition 5.3 is that the symmetric group action on LR tableaux via crystal reflection operators translates to the usual permutation action on the parts of the shape underlying the rectification, after applying the  $\rho$  map.

**5.1. Revisiting the LR rules in [7].** To conclude this article, we briefly describe an equivalent interpretation of our central result in the language of box-adding operators on compositions. By Proposition 4.6, we know that elements of  $\mathbf{LRT}^\sigma(\lambda, \mu, \nu)$  may be constructed by computing  $\lambda/\mu$ -compatible words  $w$  satisfying  $\mathbf{colform}(w) = (w_0\sigma w_0) \cdot \nu^r$ , where  $w_0$  is the longest word in  $\mathfrak{S}_{\ell(\nu)}$ . Equivalently, these words are exactly the growth words of standardizations of tableaux in  $\mathbf{LRT}^\sigma(\lambda, \mu, \nu)$ .

Thus, they may be identified with certain saturated chains in Young's lattice from  $\mu$  to  $\lambda$ . By applying the map  $\rho_\beta$  where  $\mathbf{sort}(\beta) = \mu$ , these chains may be interpreted as chains in  $\mathcal{L}_c$  from  $\beta$  to certain compositions  $\gamma$  that satisfy  $\mathbf{sort}(\gamma) = \lambda$ . Fixing a  $\gamma$  and counting these chains allows us to compute  $C_{\alpha\beta}^\gamma$  where  $\alpha = \sigma \cdot \nu$ .

We make the preceding discussion precise by phrasing the result in the language of box-adding operators on compositions introduced in [39] following the seminal work of Fomin [10]. Given  $i \geq 1$  and a composition  $\alpha$  that has at least one part equaling  $i-1$ , we define  $\mathbf{t}_i(\alpha)$  to be the unique composition covering  $\alpha$  in  $\mathcal{L}_c$  where the new box occurs in the  $i$ -th column. Given a word  $w = w_1 \dots w_n$ , define  $\mathbf{t}_w := \mathbf{t}_{w_1} \cdots \mathbf{t}_{w_n}$ . We have the following corollary.

**Corollary 5.4.** *Let  $\alpha, \beta$  and  $\gamma$  be compositions such that  $|\beta| + |\alpha| = |\gamma|$ . Let  $(\lambda, \mu, \nu)$  be the triple  $(\mathbf{sort}(\gamma), \mathbf{sort}(\beta), \mathbf{sort}(\alpha))$ . If  $\sigma \in \mathfrak{S}_{\ell(\nu)}$  satisfies  $\alpha = \sigma \cdot \nu$ , then*

$$C_{\alpha\beta}^\gamma = |\{w \in \mathbf{Frank}(\lambda, \mu, (w_0\sigma w_0) \cdot \nu^r) \mid \mathbf{t}_w(\beta) = \gamma\}|.$$

The cases in Corollary 5.4 where  $\sigma$  is either the identity or the longest permutation in  $\mathfrak{S}_{\ell(\nu)}$  correspond to left and right LR rules of [7]. We reiterate here that the two LR rules in [7] were proved by different approaches. Thus Corollary 5.4 generalize these LR rules by providing a uniform proof that works in all cases.

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