QUIZ 1 Solutions

(8) 1. Give the definitions:
   a. A sequence \( \{a_n\}_{n=1}^\infty \) is convergent iff …
   there is a real number \( A \) such that for every \( \varepsilon > 0 \) there exists an integer \( N \) such that if \( n \geq N \), then \( |a_n - A| < \varepsilon \).
   b. Let \( S \) be a set of real numbers. A real number \( A \) is an accumulation point of \( S \) iff …
   (For this part only: full credit for any statement equivalent to the definition.)
   every neighborhood of \( A \) contains infinitely many points of \( S \).
   c. State the Least Upper Bound Property of \( \mathbb{R} \).
   Every non-empty subset of \( \mathbb{R} \) that is bounded from above has a least upper bound.
   d. State the Bolzano-Weierstrass Theorem, which concerns sets of real numbers that have accumulation points.
   Every bounded and infinite set of real numbers has at least one accumulation point.

(6) 2. Suppose \( E \subset \mathbb{R} \) is non-empty and that \( E \cap [0, 1] = \emptyset \).
   a. Is it possible that \( \text{sup } E = 0 \)? If “yes”, give an example of such a set. If “no”, explain why not.
      Yes. An example is the set \( E = (-1, 0) \).
   b. Is it possible that \( \text{sup } E = 1 \)? If “yes”, give an example of such a set. If “no”, explain why not.
      No. If \( \text{sup } E = 1 \), then from hw #0.44 \( (1 - \varepsilon, 1] \cap E \neq \emptyset \) for all \( \varepsilon > 0 \). But \( (0, 1] \cap E = [0, 1] \cap E = \emptyset \), from the hypothesis, so taking \( \varepsilon = 1 \) in hw #0.44 gives a contradiction.

(4) 3. Give an example of a set \( S \) of real numbers that has exactly two accumulation points, 0 and 1.
   An example is
   \[ S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \left\{ 1 + \frac{1}{n} : n \in \mathbb{N} \right\}. \]
   Compare to hw #1.22, and examples from your class notes.

(7) 4. Prove that every convergent sequence is a Cauchy sequence. (This is a theorem in the text. Don’t just refer to another theorem coming after this in the book; give a proof based
on the definitions, using $\varepsilon > 0$.)

This is Theorem 1.3 in the text. The proof is a “standard $\varepsilon/2$ argument”, which you can find in the text.