Here are some problems taken from various sources, including my head. Some are more difficult than the typical problems in the text. You are not expected to do all (but it’s ok if you do!), but please do some. There is no “due date”. Please turn them in as you complete them, not all in one batch.

1. Suppose that $S$ is a family of holomorphic functions on a domain $G$ and that $S$ is equicontinuous at each point of $G$. Show that $S$ is equicontinuous on every compact subset of $G$.

2. (See Sarason #X.19.5) Let $G = \{ z : |\text{Im} \, z | < 1 \}$ and let $f \in H(G)$ be bounded and such that $\lim_{x \to +\infty} f(x) = 0$. Prove that if $0 < r < 1$, then $\lim_{x \to +\infty} f(x + iy) = 0$ uniformly for $y \in [-r, r]$.

3. Let $p_n(z) = \sum_{k=0}^{n} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$. Use complex methods to prove that $\pi$ is in the closure of the set $\bigcup_{n=1}^\infty \{ z : p_n(z) = 0 \}$.

4. Suppose $f$ is holomorphic on a region containing the closed disk $\{ z : |z| \leq 1 \}$ and $|f(z)| = 1$ when $|z| = 1$. Suppose $f$ has a zero of order 2 at $z = 1/2$. Can $f(0) = 1/2$? Prove it.

5. a. (A Minimum Principle) Prove that if $f$ is a non-constant analytic function on a bounded open set $G$ and is continuous on the closure of $G$, then either $f$ has a zero in $G$ or $|f|$ assumes its minimum value on $\partial G$.

   b. Let $G$ be a bounded and connected open set and suppose $f \in H(G)$ is continuous on the closure of $G$. Show that if $|f|$ is constant on $\partial G$, then either $f$ is constant on $G$ or $f$ has a zero in $G$.

6. Let $f \in H(B(0, 1))$. Prove that the integral $\int_{0}^{2\pi} |f(re^{i\theta})|d\theta$ is a strictly increasing function of $r$, unless $f$ is constant. [Hint: For $0 < r_1 < r_2 < 1$, define the function $A$ so that $A(\theta)f(r_1 e^{i\theta}) = |f(r_1 e^{i\theta})|$, $0 \leq \theta \leq 2\pi$. Apply the Maximum Modulus Principle to the function $F(z) = \int_{0}^{2\pi} A(\theta)f(z e^{i\theta})d\theta$ on the disk $B(0, r_2)$].

7. Suppose $f \in H(B(0, 1))$. Prove that there is a sequence $\{ z_n \} \subset B(0, 1)$ such that $|z_n| \to 1$ and $\{ f(z_n) \}$ is bounded.
8. Let $F \subset H(G)$ be a normal family and $\{f_n\} \subset F$. Suppose that no $f_n$ takes the value 0 and that $f_n(z_0) \to 0$ for some $z_0 \in G$. Prove that $f_n \to 0$ locally uniformly in $G$.

9. Let $G$ be a bounded domain with $z_0 \in G$. Prove that among all $f \in H(G)$ that are bounded in modulus by 1 and vanish at $z_0$, there is one (at least) that maximizes the integral $\iint_{G} |f(x + iy)| \, dx \, dy$.

10. Suppose $G \subseteq \mathbb{C}$ is simply connected and symmetric with respect to the real axis, and that $0 \in G$. Prove that if $f$ is the Riemann mapping function from $B(0, 1)$ onto $G$ with $f(0) = 0$ and $f'(0) > 0$, then $\text{Im} \, f(x) = 0$ for $-1 < x < 1$.

(Hint: Use the observation that if $g \in H(\Omega)$, then $g^*(z) = \overline{g(\overline{z})} \in H(\Omega^*)$, where $\Omega^*$ is the reflection of $\Omega$ across the real axis. With $f$ as given in the problem, what can you say about $f^*$?)