AN EXTENSION OF THE SCHWARZ–PICK LEMMA

H. S. BEAR AND WAYNE SMITH

Abstract. The Schwarz–Pick lemma states that an analytic map of the disk (or half-plane) into itself decreases all hyperbolic distances, and if one distance is preserved, then the mapping is an isometry. The Harnack metric is defined in quite general domains, and coincides with the hyperbolic metric in the disk and half-plane. We prove that an analytic map $\varphi$ of the disk or half-plane into a general domain $G$ decreases all Harnack distances, and if one distance is preserved then either $G$ is simply connected and $\varphi$ is an isometry, or $G$ conformally equivalent to the punctured disk and $\varphi$ is a covering map.

1. Introduction

The half–plane $\mathbb{H} = \{ z : \text{Re} z > 0 \}$ is a model for hyperbolic geometry. The hyperbolic lines in $\mathbb{H}$ are the horizontal half–lines, and the arcs of circles which are orthogonal to the $y$–axis. The hyperbolic metric in $\mathbb{H}$ is given by $d\rho_{\mathbb{H}}(z) = |dz|/\text{Re} z$, so the function $\log x$ is a hyperbolic ruler on each horizontal line $\text{Im} z = \text{constant}$.

The Schwarz–Pick lemma is usually stated for the disk $D = \{ z : |z| < 1 \}$, but it will be more convenient for us to work in the half–plane $\mathbb{H}$. In $\mathbb{H}$ the Schwarz–Pick lemma states that an analytic map of $\mathbb{H}$ into $\mathbb{H}$ decreases (≤) all hyperbolic distances, and if the distance between one pair of points is preserved, then the mapping is an isometry onto $\mathbb{H}$. An immediate consequence is that the hyperbolic
metric has the important property of conformal invariance, i.e. if \( \varphi \) is a conformal automorphism of \( \mathbb{H} \), then \( \rho_H(z_1, z_2) = \rho_H(\varphi(z_1), \varphi(z_2)) \) for all \( z_1, z_2 \in \mathbb{H} \).

We will prove an extension of Schwarz–Pick for maps from \( \mathbb{H} \) to a general domain \( G \), using the Harnack metric (definition below). This metric is defined in quite general domains, and is identical to the hyperbolic metric in \( \mathbb{H} \) (or \( \mathbb{D} \)).

2. The Harnack Metric

The Harnack metric is a conformally invariant metric defined on any domain \( G \) which has enough positive harmonic functions to separate points. Domains such as \( \mathbb{C} \setminus \{0\} \) must be excluded, as it is well known that the only positive harmonic functions on this domain are constant functions; see [3, Corollary 3.3]. The condition on \( G \) that assures there are sufficiently many positive harmonic functions is that its complement has positive logarithmic capacity; see [10, Chapter 8]. We make the standing assumption that all domains considered in this paper have this property. Technical aspects of this condition will not be important in this paper; not much will be lost if the reader simply assumes that \( G \) is bounded or conformally equivalent to a bounded domain.

Let \( S \) denote the positive harmonic functions on such a domain \( G \), and let \( S_0 = S \cap \{ u : u(z_0) = 1 \} \), where \( z_0 \) is some point of \( G \). Then the Harnack metric \( d_G \) is defined by

\[
(1) \quad d_G(z_1, z_2) = \sup \left\{ |\log u(z_1) - \log u(z_2)| : u \in S \right\}
\]

\[
(2) \quad = \sup \left\{ |\log [u(z_1)/u(z_2)]| : u \in S \right\}.
\]
The sup in (1) is of course the same as the sup for \( u \in S_0 \). The extremal functions which determine distance on the horizontal lines in \( \mathbb{H} \) are the multiples of \( \text{Re} \, z \) and multiples of \( \text{Re} \, 1/(z - iy_0) \).

The Harnack metric first arose as a metric on the Gleason parts of a function algebra [4]. The appropriate name “Harnack metric” is apparently due to König [8]. The relationship between the hyperbolic metric in \( \mathbb{D} \) and the Harnack metric in \( \mathbb{D} \) is treated in detail in [5].

The set \( S_0 \) is compact in the topology of uniform convergence on compact sets in \( G \). So the sup in (1) is a max, and the max occurs at an extreme point of the compact convex set \( S_0 \). The max of course also occurs at all positive multiples of such an extreme point. These extreme rays in \( S \), the positive multiples of the extreme points of \( S_0 \), are the minimal positive harmonic functions of R. S. Martin [9]. Such a minimal function \( u \) is characterized by the following condition: if \( 0 < v \leq u \), then \( v = cu \) for some positive constant \( c \). [5]

We observe that if \( \varphi \) is an analytic function from a domain \( G_1 \) to a domain \( G_2 \), then \( \varphi \) decreases (\( \leq \)) all Harnack distances. This is because if \( u \) is a positive harmonic function on \( G_2 \), then \( u \circ \varphi \) is a positive harmonic function on \( G_1 \). Again, we immediately get conformal invariance: if \( \varphi : G_1 \rightarrow G_2 \) is one-to-one and onto, and \( z, w \in G \), then \( d_{G_1}(z, w) = d_{G_2}(\varphi(z), \varphi(w)) \). In general, an analytic map from \( \mathbb{H} \) to a domain \( G \) decreases all Harnack distances. Our main result is this: if \( \varphi : \mathbb{H} \rightarrow G \) and \( \varphi \) preserves the distance between one pair of distinct points, then \( \varphi \) is a conformal map of \( \mathbb{H} \) onto \( G \) and hence an isometry, unless \( G \) is the punctured disk \( \mathbb{D} \setminus \{0\} \), or its conformal equivalent.
3. The Exceptional Case

We first describe the Harnack metric in $\mathbb{D} \setminus \{0\}$. A positive harmonic function on $\mathbb{D} \setminus \{0\}$ has the following form (Bôcher’s theorem, [3, Theorem 3.9]):

$$u(z) = v(z) + b \log |1/z|,$$

where $v$ is a non-negative harmonic function on $\mathbb{D}$, and $b$ is a non-negative constant. For such a function to be minimal, we must have $u(z) = v(z)$ and $u$ is a minimal function on $\mathbb{D}$, or $u(z) = b \log |1/z|$. Thus we have the following characterization of the Harnack metric in $\mathbb{D} \setminus \{0\}$:

3.1. **Lemma.** Denote by $d_0$ be the Harnack metric on $\mathbb{D} \setminus \{0\}$. If $z, w \in \mathbb{D} \setminus \{0\}$, then

$$d_0(z, w) = \max \{d_\mathbb{D}(z, w), |\log \log |1/z| - \log \log |1/w| |\}.$$

3.2. **Example.** The function $g(z) = \exp(-z)$ maps $\mathbb{H}$ onto $\mathbb{D} \setminus \{0\}$ and preserves Harnack distances on each horizontal line in $\mathbb{H}$. All other distances strictly decrease.

**Proof.** We use “horizontal line” in $\mathbb{H}$ to mean the hyperbolic line, which is the Euclidean half-line $x > 0$. Each horizontal line in $\mathbb{H}$ is mapped one-to-one onto a radius in $\mathbb{D} \setminus \{0\}$. Let $z_1 = x_1 + iy_0$ and $z_2 = x_2 + iy_0$, so that $d_\mathbb{H}(z_1, z_2) = |\log x_1 - \log x_2|$. By Lemma 3.1 we know that $d_0(g(z_1), g(z_2))$ is the larger of $d_\mathbb{D}(g(z_1), g(z_2))$ and $|\log \log |1/g(z_1)| - \log \log |1/g(z_2)| |$. Using the fact that $|g(z_1)| = e^{-x_1}$ and $|g(z_2)| = e^{-x_2}$, we get

$$d_0(g(z_1), g(z_2)) \geq |\log \log e^{x_1} - \log \log e^{x_2}| = d_\mathbb{H}(z_1, z_2).$$
Since any mapping decreases distances, \( d_0(g(z_1), g(z_2)) = d_\mathbb{H}(z_1, z_2) \). Notice that \( d_0(g(z_1), g(z_2)) \) could not also be equal to \( d_\mathbb{D}(g(z_1), g(z_2)) \), for then \( g \) would be a map of \( \mathbb{H} \) into \( \mathbb{D} \) which preserves a distance. By Schwarz–Pick \( g \) would have to be a conformal map of \( \mathbb{H} \) onto \( \mathbb{D} \), which it obviously is not. Now consider \( z_1, z_2 \in \mathbb{H} \) that do not lie on the same horizontal line. As before, either \( d_0(g(z_1), g(z_2)) = d_\mathbb{D}(g(z_1), g(z_2)) \) or \( d_0(g(z_1), g(z_2)) = d_\mathbb{D}(\text{Re } z_1, \text{Re } z_2) \). In the first case we use the fact that \( d_\mathbb{D}(g(z_1), g(z_2)) < d_\mathbb{H}(z_1, z_2) \) by Schwarz–Pick as just seen, while in the second case we use the fact that \( d_\mathbb{H}(\text{Re } z_1, \text{Re } z_2) < d_\mathbb{H}(z_1, z_2) \) for every pair \( w_1, w_2 \) in \( G \) there is always a

4. The Hyperbolic Metric in General Domains

To show that \( \mathbb{D} \setminus \{0\} \) is the only domain for which the extended Schwarz–Pick lemma fails, we need some properties of the hyperbolic metric in general domains.

The hyperbolic metric in a general domain \( G \) is defined in terms of a universal covering map of \( G \). Every plane domain whose complement has at least two points (and hence any domain with complement having positive logarithmic capacity) has a universal covering map \( \pi : \mathbb{H} \rightarrow G \), where \( \pi \) is an analytic map on \( \mathbb{H} \) onto \( G \) and every point of \( G \) has a neighborhood \( U \) such that \( \pi \) maps each component of \( \pi^{-1}(U) \) conformally onto \( U \). It follows that distinct components of \( \pi^{-1}(U) \) for such a neighborhood are conformally equivalent. Covering maps of \( G \) are unique up to automorphisms of \( \mathbb{H} \). (See [1] for details of the Uniformization Theorem.) The hyperbolic distance \( \rho_G(z_1, z_2) \) in \( G \) is the minimum of the hyperbolic distances between points in \( \mathbb{H} \) which map onto \( z_1 \) and \( z_2 \) respectively. Clearly \( \rho_G(\pi(z_1), \pi(z_2)) \leq \rho_\mathbb{H}(z_1, z_2) \) for all \( z_1, z_2 \in \mathbb{H} \), and for every pair \( w_1, w_2 \) in \( G \) there is always a
pair $z_1, z_2 \in H$ with $\pi(z_1) = w_1$, $\pi(z_2) = w_2$, and $\rho_H(z_1, z_2) = \rho_G(w_1, w_2)$ [7, p. 685].

The following lemma is well known application of the Monodromy Theorem; see for example [6, Theorem 16.1.3].

**4.1. Lemma 2.** Let $\pi : H \to G$ be a covering map of $G$. If $g : \Omega \to G$ is an analytic map of a simply connected domain $\Omega$ into $G$, then $g$ can be lifted to a map $\hat{g} : \Omega \to \mathbb{H}$ such that $\pi \circ \hat{g} = g$.

The next lemma for Harnack distance was observed in §2. It is well known for hyperbolic distance; see for example [7, p.685].

**4.2. Lemma.** An analytic map $\varphi : G_1 \to G_2$ decreases (≤) all Harnack and hyperbolic distances.

**4.3. Corollary.** For any domain $G$ and all $w_1, w_2 \in G$, $d_G(w_1, w_2) \leq \rho_G(w_1, w_2)$.

**Proof.** Let $\pi : \mathbb{H} \to G$ be a covering map and let $w_1, w_2 \in G$. Pick $z_1, z_2 \in \mathbb{H}$ with $\pi(z_i) = w_i$ and $\rho_G(w_1, w_2) = \rho_H(z_1, z_2)$. Since the analytic map $\pi$ decreases Harnack distances, we have

$$d_G(w_1, w_2) \leq d_H(z_1, z_2) = \rho_H(z_1, z_2) = \rho_G(w_1, w_2).$$

□

**4.4. Lemma.** If $g : \mathbb{H} \to G$ is an analytic map of $\mathbb{H}$ into $G$ and $g$ preserves Harnack or hyperbolic distance for one pair, then $g$ is a universal covering map of $\mathbb{H}$ onto $G$.

**Proof.** Suppose $z_1, z_2$ are distinct points of $\mathbb{H}$ with $d_H(z_1, z_2) = d_G(g(z_1), g(z_2))$. Let $\pi$ be a universal covering map of $G$ by $\mathbb{H}$ and let $\hat{g} : H \to \mathbb{H}$ be a lift of $g$, so
that \( \pi \circ \hat{g} = g \). Two applications of Lemma 4.2 now show that

\[
d_{\mathbb{H}}(z_1, z_2) \geq d_{\mathbb{H}}(\hat{g}(z_1), \hat{g}(z_2)) \geq d_G(\pi \circ \hat{g}(z_1), \pi \circ \hat{g}(z_2)) = d_G(g(z_1), g(z_2)).
\]

By assumption \( d_{\mathbb{H}}(z_1, z_2) = d_G(g(z_1), g(z_2)) \), and so we get equality across the display above. Since the Harnack and hyperbolic distances agree on \( \mathbb{H} \), \( \rho_{\mathbb{H}}(z_1, z_2) = \rho_{\mathbb{H}}(\hat{g}(z_1), \hat{g}(z_2)) \), and by the equality case of Schwarz-Pick we see that \( \hat{g} \) is an automorphism of \( \mathbb{H} \) and \( g = \pi \circ \hat{g} \) is a universal covering map. The proof for the hyperbolic distance is similar. \( \square \)

4.5. Theorem. If \( g : \mathbb{H} \longrightarrow G \) is an analytic map of \( \mathbb{H} \) into \( G \), and \( g \) preserves Harnack distance for a distinct pair \( a, b \in \mathbb{H} \), then \( g \) is a conformal map of \( \mathbb{H} \) onto \( G \) and hence an isometry, or \( G \) is conformally equivalent to the punctured disk \( \mathbb{D} \setminus \{0\} \), and \( g \) is a universal covering map of \( G \) by \( \mathbb{H} \).

Proof. From Lemma 4.4 we know that \( g \) is a universal covering map onto \( G \). If \( g \) is one–to–one, then \( g \) is a conformal map onto \( G \). So we assume \( g \) is not one–to–one.

We may assume by making a pre–mapping of \( \mathbb{H} \) that \( a = 1 \) and \( b = x_0 > 1 \) are the points whose distance is preserved. Let \( u \) be a positive harmonic function on \( G \) which determines the distance from \( g(1) \) to \( g(x_0) \):

\[
d_G(g(1), g(x_0)) = |\log u(g(1)) - \log u(g(x_0))|.
\]

Then \( u \circ g \) is a positive harmonic function on \( \mathbb{H} \) which determines \( d_{\mathbb{H}}(1, x_0) \). The harmonic function \( \text{Re} \, z \) (the Poisson kernel at \( \infty \)) and its positive multiples are functions which give \( d_{\mathbb{H}}(1, x_0) \). We may assume that \( u \) is chosen so that \( u \circ g(z) = \)
Re $z$ for all $z \in \mathbb{H}$. For points $z_1$, $z_2$ on any horizontal line in $\mathbb{H}$,
\[
d_{\mathbb{H}}(z_1, z_2) \geq d_G(g(z_1), g(z_2)) \geq \left| \log u \circ g(z_1) - \log u \circ g(z_2) \right| = \left| \log \text{Re } z_1 - \log \text{Re } z_2 \right| = d_{\mathbb{H}}(z_1, z_2).
\]
Hence $g$ is one–to–one and preserves all distances on every horizontal line in $\mathbb{H}$.

Since $g$ is not one–to–one, there are distinct points $z_1$, $z_2 \in \mathbb{H}$ with $g(z_1) = g(z_2) = w_0$, so $u \circ g(z_1) = u \circ g(z_2)$, and $\text{Re } z_1 = \text{Re } z_2$. Let $z_2$ be the nearest point above $z_1$ with $g(z_2) = g(z_1)$. There is such a nearest point or $z_1$ would be a limit point of points $z_n$ with $g(z_n) = g(z_1)$, and $g$ would be constant.

Since $g$ is a covering map, there is a neighborhood $\Delta$ of $w_0$ such that every component of $g^{-1}(\Delta)$ maps conformally onto $\Delta$ (and all such components are conformally equivalent to each other). Let $\Delta(z_1)$, $\Delta(z_2)$ be the components of $g^{-1}(\Delta)$ containing $z_1$, $z_2$ respectively. For $z \in \Delta(z_1)$, let $\varphi(z)$ be the point in $\Delta(z_2)$ with the same $g$–image as $z$, so $g(z) = g(\varphi(z))$. Since $u \circ g(z) = \text{Re } z$ for all $z \in \mathbb{H}$, for $z \in \Delta(z_1)$ we have
\[
u \circ g(z) = \text{Re } z = u \circ g(\varphi(z)) = \text{Re } (\varphi(z)).
\]
That is, for each $z \in \Delta(z_1)$ there is a real number $w(z)$ such that $\varphi(z) = z + iw(z)$.

Since $\varphi$ is analytic, $w(z)$ is a positive constant $w$, and $g(z) = g(z + iw)$ for all $z \in \Delta(z_1)$. The function $g(z) - g(z + iw)$ is analytic on $\mathbb{H}$, and zero on the open set $\Delta(z_1)$, so $g(z) = g(z + iw)$ for all $z \in \mathbb{H}$. The function $g$ is periodic with period $iw$, and $g$ maps the strip between the lines containing $z_1$ and $z_2$, together with one
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of the lines, one–to–one onto $G$. The mapping is one–to–one because $z_2$ is chosen closest to $z_1$.

To simplify the calculations let us assume $w = 2\pi$, so $g(z) = g(z + 2\pi i)$ for all $z$. Let $h(z) = e^{-z}$ for $z \in \mathbb{H}$. As we saw in Section 3, $h$ is a one-to-one map of the strip $0 \leq \text{Im} \, z < 2\pi$ onto the punctured disk, and $h$ preserves distances on each horizontal line. Since $g$ is a one-to-one map of the same strip onto $G$, $G$ is conformally equivalent to $\mathbb{D} \setminus \{0\}$, and distance is preserved on the curves in $G$ which map onto radii. (For a general characterization of periodic analytic functions see [2, p. 363, 364].)

4.6. Corollary. If $G$ is a domain and $d_G(w_1, w_2) = \rho_G(w_1, w_2)$ for some distinct pair $w_1, w_2 \in G$, then either

(i) $G$ is simply connected and $d_G = \rho_G$, or

(ii) $G$ is conformally equivalent to the punctured disk and if $\varphi : G \to \mathbb{D} \setminus \{0\}$ is a conformal map, then $\varphi(w_1)$ and $\varphi(w_2)$ lie on the same radius of $\mathbb{D}$.

Proof. Let $\pi : \mathbb{H} \to G$ is a covering map with $\pi(z_i) = w_i$ and $\rho_{\mathbb{H}}(z_1, z_2) = \rho_G(w_1, w_2)$. Then $d_{\mathbb{H}}(z_1, z_2) = \rho_{\mathbb{H}}(z_1, z_2) = \rho_G(w_1, w_2) = d_G(w_1, w_2)$, so $\pi$ preserves a Harnack distance. Therefore Theorem 4.5 applies to $\pi$ in place of $g$. In the case that $G$ is conformally equivalent to the punctured disk, with $\varphi : G \to \mathbb{D} \setminus \{0\}$ a conformal map, then $\varphi \circ \pi : \mathbb{H} \to \mathbb{D} \setminus \{0\}$ is a universal covering map and we saw in Example 3.2 that $\varphi(w_1)$ and $\varphi(w_2)$ lie on the same radius of $\mathbb{D}$. □

References


**University of Hawaii, Honolulu, HI 96822, USA**
*E-mail address: bear@math.hawaii.edu*

**University of Hawaii, Honolulu, HI 96822, USA**
*E-mail address: wayne@math.hawaii.edu*