SARASON'S COMPOSITION OPERATOR OVER 
THE HALF-PLANE

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In memory of Donald Sarason

Abstract. Let \( H = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \} \) be the upper half plane, and denote by \( L^p(\mathbb{R}) \), \( 1 \leq p < \infty \), the usual Lebesgue space of functions on the real line \( \mathbb{R} \). We define two “composition operators” acting on \( L^p(\mathbb{R}) \) induced by a Borel function \( \varphi : \mathbb{R} \to \mathbb{H} \), by first taking either the Poisson or Borel extension of \( f \in L^p(\mathbb{R}) \) to a function on \( \mathbb{H} \), then composing with \( \varphi \) and taking vertical limits. Classical composition operators, induced by holomorphic functions and acting on the Hardy spaces \( H^p(\mathbb{H}) \) of holomorphic functions, correspond to a special case. Our main results provide characterizations of when the operators we introduce are bounded or compact on \( L^p(\mathbb{R}) \), \( 1 \leq p < \infty \). The characterization for the case \( 1 < p < \infty \) is independent of \( p \) and the same for the Poisson and the Borel extensions. The case \( p = 1 \) is quite different.

1. Introduction

In 1990 D. Sarason [15] introduced the viewpoint of composition operators with holomorphic symbols as integral operators acting on spaces of functions defined on the unit circle \( \mathbb{T} \). Namely, for a holomorphic self-map \( \varphi \) of the unit disk \( \mathbb{D} \) of the complex plane \( \mathbb{C} \), Sarason introduced an operator which maps an integrable function \( f \) on \( \mathbb{T} \) to a function defined on \( \mathbb{T} \) by taking the Poisson transform of \( f \), composing with \( \varphi \), and then taking radial limits. When restricted to the Hardy spaces, every such operator turns out to coincide with the classical composition operator with symbol \( \varphi \). Also, every such operator turns out to be \( L^1 \)-bounded, and problems such as characterizing when the operator is \( L^1 \)-compact were studied by Sarason.

In a recent paper [3] the current authors extended Sarason’s idea to the setting of higher dimensional complex balls and generalized it in two significant ways. First, four natural integral transforms including the Poisson transform are considered. Second, holomorphic symbol functions were extended to general ones. In this paper we study the analogues of such generalizations of Sarason’s idea in the setting of the half-plane.

Due to the half-plane \( \mathbb{H} := \{ z \in \mathbb{C} : \text{Im} \, z > 0 \} \) being unbounded, there are significant differences between the theory of composition operators acting in that setting and the theory in the setting of complex balls. For example, in the setting of complex balls composition operators induced by constants are bounded, and even
compact. In contrast, a composition operator induced by a constant symbol is unbounded in the setting of $H$, since non-zero constant functions are not integrable over $R$. More generally, the holomorphic symbols of classical composition operators that are bounded on the Hardy spaces of $H$ are characterized by the rigid requirements that they must map infinity to infinity and have finite angular derivative there, and none of these operators are compact; see [12] or [8]. The generalized composition operators we define are not subject to such rigid requirements. Rather, the theory we develop parallels the theory in [3] of the corresponding operators on complex balls; in particular there are rich families of bounded and compact operators. We note, however, that even though the theory in the setting of $H$ is similar to that on complex balls, new methods are required due to $H$ being unbounded.

We begin with some background needed to define the operators. We denote by $S(H)$ the class of all Borel functions

$$\phi : R \to \overline{H} \text{ such that } \mu_{\phi}|_R \ll m.$$  

\begin{equation}
\label{eq:1.1}
\end{equation}

Here and throughout the paper, $m$ is the Lebesgue measure on the real line $R$, $\mu_{\phi}|_R$ is the restriction to $R$ of the pullback measure $\mu_{\phi} := m \circ \phi^{-1}$ defined by $\mu_{\phi}(E) := m[\phi^{-1}(E)]$ for Borel sets $E \subset H$. This pullback measure naturally appears in the measure theoretic change of variables; see Section 2.5.

We consider two natural integral transforms against the reproducing kernels $K^b$ and $K^h$ defined by

$$K^b(z, t) := \frac{1}{2\pi i} \cdot \frac{1}{t - z} \quad (i = \sqrt{-1})$$

and

$$K^h(z, t) := K^b(z, t) + \overline{K^b(z, t)} = \frac{1}{\pi} \cdot \frac{\text{Im } z}{|z - t|^2}$$

for $(z, t) \in H \times R$. The kernel $K^b$, often referred to as the Borel kernel (see Section 2.3), is the reproducing kernel of the Hardy spaces over $H$ (see [7, Theorem 11.8]) and $K^h$ is the well-known Poisson kernel. Associated with these kernels are the Borel transform or the Poisson transform given by

$$f^\sigma(z) := \int_R f(t)K^\sigma(z, t) \, dt, \quad z \in H$$

\begin{equation}
\label{eq:1.2}
\end{equation}

for functions $f \in L^p(R)$, $1 \leq p < \infty$, and $\sigma \in \{b, h\}$.

As is well known, each function $f \in L^p(R)$, $1 \leq p \leq \infty$, is recovered $m$-a.e. by the vertical limits of its Poisson transform, i.e.,

$$\lim_{y \downarrow 0} f^h(x + yi) = f(x)$$

\begin{equation}
\label{eq:1.3}
\end{equation}

at $m$-almost every $x \in R$. Also is well known that $f^b$ has vertical limits at $m$-almost every point of $R$; see Section 2.3. We now use these vertical limits to extend the definition of $f^\sigma$ up to the boundary. More explicitly, we define

$$f^\sigma(z) := \lim_{y \downarrow 0} f^\sigma(z + yi), \quad z \in \overline{H}$$

\begin{equation}
\label{eq:1.4}
\end{equation}

for $\sigma \in \{b, h\}$. Note that $f^\sigma$ is holomorphic or harmonic on $H$, but is defined $m$-a.e. on $R$. We also note that when $f$ is real valued, $f^h$ corresponds to the imaginary part of the classical Hilbert transform of $f$; see Section 2.3.
For \( \varphi \in S(H) \) and \( \sigma \in \{b, h\} \), we now define Sarason’s composition operator \( S^\sigma_\varphi \) with symbol \( \varphi \) by

\[
S^\sigma_\varphi f := f^\sigma \circ \varphi
\]

for functions \( f \in L^p(R) \), \( 1 \leq p < \infty \). This is clearly well defined, because \( f^\sigma \) remains the same even if \( f \) is altered on an \( m \)-null set. Also, it should be remarked that this defines \( S^\sigma_\varphi \) off an \( m \)-null set on \( R \). To see this, notice that from (1.4) we have

\[
S^\sigma_\varphi f(x) = \lim_{y \downarrow 0} f^\sigma(\varphi(x) + yi), \quad x \in R,
\]

and this limit exists precisely when \( f^\sigma \) has a vertical limit at \( \varphi(x) \). Thus, denoting by \( E \subset R \) the \( m \)-null set where \( f^\sigma \) fails to have a vertical limit, we see that \( S^\sigma_\varphi f \) has been defined at points \( x \in R \setminus \varphi^{-1}(E) \). Since \( E \) is a \( \mu_\varphi \)-null set by the assumption (1.1) of absolute continuity, \( S^\sigma_\varphi f \) has been defined \( m \text{-a.e.} \) on \( R \). When \( \varphi \) is the boundary function of a holomorphic self-map \( H \), both \( S^b_\varphi \) and \( S^h_\varphi \) acting on the Hardy spaces coincide with the classical composition operator associated with \( \varphi \); see Section 2.6.

Our first result is to characterize \( L^1 \)-boundedness/compactness of the operators. To state it we introduce an auxiliary function. Namely, using the extended kernels described in Section 2.1, we define

\[
\Lambda^\sigma_{\varphi, 1}(z) := \|\mathcal{K}^\sigma(z, \varphi(\cdot))\|_1, \quad z \in \hat{H}
\]

where

\[
\hat{H} := \overline{H} \cup \{\infty\}
\]

is the one-point compactification of \( H \). Here and in what follows, \( \| \cdot \|_p \) denotes the \( L^p \)-norm with respect to the measure \( m \) on \( R \). This function is well defined, because each \( \mathcal{K}^\sigma(z, \varphi(\cdot)) \) with \( z \in \mathbb{R} \) is a Borel function defined on \( R \) off the \( m \)-null set \( \varphi^{-1}(\{z\}) \). Also, note \( \Lambda^\sigma_{\varphi, 1}(\infty) = 0 \), because \( \mathcal{K}^\sigma(\infty, \cdot) = 0 \) by definition.

The next theorem is our characterization for \( p = 1 \). In fact the space \( L^1(R) \) in the next theorem can be expanded to the space of all complex Borel measures on \( R \); see Theorems 3.1 and 3.9.

**Theorem 1.1.** Let \( \varphi \in S(H) \) and \( \sigma \in \{b, h\} \). Then the following statements hold:

(a) \( S^\sigma_\varphi \) is bounded on \( L^1(R) \) if and only if \( \Lambda^\sigma_{\varphi, 1} \) is bounded on \( H \);

(b) \( S^\sigma_\varphi \) is compact on \( L^1(R) \) if and only if \( \Lambda^\sigma_{\varphi, 1} \in C(\hat{H}) \).

In case of holomorphic symbols (more precisely, symbols that are boundary functions of holomorphic self-maps of \( H \)), we obtain two additional \( L^1 \)-characterizations for the operators associated with the Poisson kernel; see Theorems 5.1 and 5.5 in Section 5. In this regard, we mention an interesting comparison to the theory with holomorphic symbols on the disk. In contrast to the setting of \( H \), boundedness was not an issue in the setting of the disk, since the disk analogue of \( S^h_\varphi \) is bounded on \( L^1(T) \) for every holomorphic \( \varphi \). The comparison regarding compactness, on the other hand, is reversed. In the setting of the half-plane, \( S^h_\varphi \) is not compact on \( L^1(R) \) for any holomorphic symbol \( \varphi \), while \( L^1(T) \)-compactness of its disk analogue was an interesting question studied in [15] by Sarason in the setting of the disk.

Our second result is to characterize \( L^p \)-boundedness/compactness, \( 1 < p < \infty \), of the operators. It turns out that the characterizations for \( 1 < p < \infty \) are quite
different from the case \( p = 1 \). We also need auxiliary functions \( \Lambda_{\sigma,p} \), depending on \( p \), defined by

\[
\Lambda_{\sigma,p}(z) := \frac{\|\mathcal{K}_{\sigma}(z,\varphi(\cdot))\|_p}{\|K_{\sigma}(z,\cdot)\|_p}, \quad z \in \mathbf{H}.
\]

The next theorem is our characterization for \( 1 < p < \infty \). In fact boundedness/compactness of \( S_{\varphi}^{\sigma} \) on \( L^p(\mathbf{R}) \) is independent of \( \sigma \in \{b,h\} \) and \( p \in (1,\infty) \); see the more detailed Theorems 4.5 and 4.6. In what follows we denote by \( C_0(\hat{\mathbf{H}}) \) the class of all continuous functions on \( \mathbf{H} \) that continuously extend to \( \hat{\mathbf{H}} \) and vanish on \( \partial\hat{\mathbf{H}} = \mathbf{R} \cup \{\infty\} \).

**Theorem 1.2.** Let \( \varphi \in \mathcal{S}(\mathbf{H}) \), \( \sigma \in \{b,h\} \) and \( 1 < p < \infty \). Then the following statements hold:

(a) \( S_{\varphi}^{\sigma} \) is bounded on \( L^p(\mathbf{R}) \) if and only if \( \Lambda_{\varphi,p}^{\sigma} \) is bounded on \( \mathbf{H} \);

(b) \( S_{\varphi}^{\sigma} \) is compact on \( L^p(\mathbf{R}) \) if and only if \( \Lambda_{\varphi,p}^{\sigma} \in C_0(\hat{\mathbf{H}}) \).

Note that the operator \( S_{\varphi}^{b} \) has natural several-variable extensions based on the Poisson kernels of the higher dimensional half-spaces. We remark that our results above for the case \( \sigma = h \) also extend to such higher dimensional settings by essentially the same arguments.

The rest of the paper is organized as follows: In Section 2 we collect some general background material that we need later. In Section 3 we prove expanded versions of \( L^1 \)-characterizations including the action of the operators on the space of all complex Borel measures on \( \mathbf{R} \); see Theorems 3.1 and 3.9. We also observe several consequences and exhibit related examples. In Section 4 we prove more detailed versions of \( L^p \)-characterizations; see Theorems 4.5 and 4.6. Also, applying our characterizations, we show that \( L^1 \)-boundedness/compactness implies \( L^p \)-boundedness/compactness; see Corollary 4.2. However, the converse of this implication fails; an explicit example is included.

In Section 5 consideration is restricted to symbols \( \varphi \) given by the boundary function of a holomorphic self map of \( \mathbf{H} \). For such \( \varphi \), we give two alternate characterizations of when \( S_{\varphi}^{b} \) is bounded on \( L^1(\mathbf{R}) \) and show that this happens if and only if the ordinary composition operator \( C_{\varphi} \) is bounded on the Hardy space \( \mathcal{H}^1(\mathbf{H}) \).

**Constants.** Throughout the paper we use the same letter \( C \) to denote positive constants which may vary at each occurrence but do not depend on the essential parameters. Variables indicating the dependency of constants \( C \) will be sometimes specified inside parentheses. For nonnegative quantities \( X \) and \( Y \) the notation \( X \lesssim Y \) or \( Y \gtrsim X \) means \( X \leq CY \) for some inessential constant \( C \). Similarly, we write \( X \approx Y \) if both \( X \lesssim Y \) and \( Y \lesssim X \) hold.

2. Preliminaries

In this section we collect some basic notions and related facts for later use.

2.1. Extended Kernels. For each \( w \in \mathbf{H} \), we denote by \( \mathcal{K}^b(\cdot,w) \) the extended Borel kernel, which is the Borel transform of \( K^b(w,\cdot) \). By the reproducing property of the Borel kernel, we have

\[
\mathcal{K}^b(z,w) = \frac{1}{2\pi i} \cdot \frac{1}{w - z}.
\]
for \( z, w \in \mathbb{H} \). Similarly, we denote by \( \mathcal{K}^h(\cdot, w) \) the extended Poisson kernel, which is the Poisson transform of \( K^h(w, \cdot) \). It is easily checked that
\[
\mathcal{K}^h(z, w) = \frac{1}{\pi} \cdot \frac{\text{Im} z + \text{Im} w}{|z - \overline{w}|^2}
\]
for \( z, w \in \mathbb{H} \).

Let \( \sigma \in \{b, h\} \). Note that \( \mathcal{K}^\sigma \) continuously extends to the points \((z, w)\in\overline{\mathbb{H}} \times \overline{\mathbb{H}}\) with \( z \neq \overline{w} \). Thus, defining \( \mathcal{K}^\sigma(z, \infty) := 0 \) and \( \mathcal{K}^\sigma(\infty, w) := 0 \) for \( z, w \in \mathbb{H} \), we see that \( \mathcal{K}^\sigma \) is now defined and continuous on \((\hat{\mathbb{H}} \times \hat{\mathbb{H}}) \setminus \Delta\). Here, \( \Delta \) denotes the diagonal of \( \hat{\mathbb{R}} \times \hat{\mathbb{R}} \) where \( \hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\} \). We will use the same notation \( \mathcal{K}^\sigma \) for such a continuous extension.

We mention some simple but useful properties for easier references later. First, we note
\[
\mathcal{K}^\sigma(z + iy, w) = \mathcal{K}^\sigma(z, w + iy)
\]
for \( y > 0 \) and \( z, w \in \mathbb{H} \). Next, we note
\[
\mathcal{K}^h(z, w) = 2\text{Re} \left[ \mathcal{K}^b(z, w) \right]
\]
for \((z, w) \in (\overline{\mathbb{H}} \times \overline{\mathbb{H}}) \setminus \Delta\). Finally, setting
\[
P_w := K^h(w, \cdot)
\]
for \( w \in \mathbb{H} \), we note
\[
P^\sigma_w(z) = \mathcal{K}^\sigma(z, w)
\]
for \( z \in \overline{\mathbb{H}} \). In fact, for \( \sigma = h \), this holds by definition. Meanwhile, for \( \sigma = b \), this holds by the fact that \( P^b_w(z) \) is precisely the Poisson transform at \( w \) of the holomorphic function \( K^b(z, \cdot) \).

2.2. Harmonic Hardy spaces. For \( 1 \leq p \leq \infty \), the harmonic Hardy space \( h^p(\mathbb{H}) \) is defined to be the Banach space of all harmonic functions \( u \) on \( \mathbb{H} \) such that
\[
\|u\|_{h^p} := \sup_{y>0} \|u_y\|_p < \infty
\]
where \( u_y(x) = u(x + yi) \). Note that the well-known \( L^p \)-Hardy space over \( \mathbb{H} \), usually denoted by \( H^p(\mathbb{H}) \), is the closed subspace of \( h^p(\mathbb{H}) \) consisting of holomorphic functions in \( h^p(\mathbb{H}) \).

As is well known, the harmonic Hardy spaces can be isometrically identified with the space of all complex Borel measures on \( \mathbb{R} \) or the Lebesgue spaces over \( \mathbb{R} \) by means of the Poisson transform \( P \). To be more precise, let \( M(\mathbb{R}) \) be the Banach space of all complex Borel measures normed by the total variation norm \( \|\cdot\|_M \).

Then
\[
P : M(\mathbb{R}) \to h^1(\mathbb{H}) \quad \text{is a surjective isometry}
\]
and
\[
P : L^p(\mathbb{R}) \to h^p(\mathbb{H}) \quad \text{is a surjective isometry}
\]
for \( 1 < p \leq \infty \); see, for example, [1, Theorem 7.17].
2.3. Borel transform. For \( f \in L^p(\mathbb{R}) \), \( 1 \leq p < \infty \), the function

\[
f^b(z) := \int_{-\infty}^{\infty} f(t) K^b(z, t) \, dt, \quad z \in \mathbb{H}
\]

is holomorphic on \( \mathbb{H} \) and is often called the Borel transform of \( f \) in the literature; see [6, Section 3.8]. As it turns out, the limit

\[
\lim_{y \downarrow 0} \int_{-\infty}^{\infty} f(t) \Im[K^b(x + yi, t)] \, dt
\]

exists at \( m \)-almost every \( x \in \mathbb{R} \) for \( f \in L^1(\mathbb{R}) \) and the map taking \( f \in L^1(\mathbb{R}) \) to the limit above turns out to be the well-known Hilbert transform (up to a constant factor); see [6, Section 3.8]. Thus, when \( 1 < p < \infty \), it follows from \( L^p \)-boundedness of the Hilbert transform that the Borel transform takes \( L^p(\mathbb{R}) \) boundedly into (actually onto) \( H^p(\mathbb{H}) \); this fails for \( p = 1 \), since the Hilbert transform is not bounded on \( L^1(\mathbb{R}) \).

Let \( H^p(\mathbb{R}) \), \( 1 \leq p < \infty \), be the space of all functions \( f \in L^p(\mathbb{R}) \) such that

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z} \, dt = 0 \quad \text{for} \quad \Im z < 0.
\]

For \( f \in H^p(\mathbb{R}) \), the above integral with \( \Im z > 0 \) represents a function in \( H^p(\mathbb{H}) \) whose boundary function is \( f \); see [7, Section 11.4]. It follows that the spaces \( H^p(\mathbb{R}) \) and \( H^p(\mathbb{H}) \) are isometrically identified via the Borel (or Poisson) transform and the vertical limiting process.

2.4. Carleson measures. For \( 1 < p < \infty \), put

\[
L^p_b(\mathbb{H}) := H^p(\mathbb{H}) \quad \text{and} \quad L^p_h(\mathbb{H}) := h^p(\mathbb{H}).
\]

We will freely identify functions in \( L^p_b(\mathbb{H}) \) and \( L^p_h(\mathbb{H}) \) with their boundary functions in \( H^p(\mathbb{R}) \) and \( L^p(\mathbb{R}) \), respectively.

Let \( \sigma \in \{b, h\} \) and \( \mu \) be a positive Borel measure on \( \mathbb{H} \). We say that \( \mu \) is a \( L^p_{\sigma} \)-Carleson measure if there exists some constant \( C > 0 \) such that

\[
\int_{\mathbb{H}} |f|^p \, d\mu \leq C \|f\|_{L^p_{\sigma}}, \quad f \in L^p_{\sigma}(\mathbb{H}).
\]

That is, \( \mu \) is an \( L^p_{\sigma} \)-Carleson measure if and only if the embedding \( L^p_{\sigma}(\mathbb{H}) \subset L^p(\mu) \) is bounded. If, in addition, this embedding is compact, then we say that \( \mu \) is a compact \( L^p_{\sigma} \)-Carleson measure. Clearly, \( h^p \)-Carleson measures are all \( H^p \)-Carleson measures. The converse also holds by the fact that

\[
\|f\|_p \leq C \|\Re f\|_p, \quad f \in H^p(\mathbb{R})
\]

for some \( C = C(p) > 0 \), which is a consequence of \( L^p \)-boundedness of the Hilbert transform. Thus the notions of \( L^p_{\sigma} \)-Carleson measures do not depend on \( \sigma \).

We now recall characterizations for \( h^p \)-Carleson measures in terms of the Carleson sets

\[
D_\delta(x) := \{z \in \mathbb{H} : |z - x| < \delta\}
\]

defined for \( x \in \mathbb{R} \) and \( \delta > 0 \). Associated with these Carleson sets are two quantities

\[
M_\delta(\mu) := \sup_{x \in \mathbb{R}} \frac{\mu[D_\delta(x)]}{\delta}
\]
and
\[ M(\mu) : = \sup_{\delta > 0} M_\delta(\mu). \]

As is well known, \( h^p \)-Carleson measures are characterized as follows:

- \( \mu \) is an \( h^p \)-Carleson measure \( \iff M(\mu) < \infty \).

Moreover, when this is the case, we have
\[ M(\mu) \approx \sup \frac{\| f^h \|_{L^p(\mu)}^p}{\| f \|_{h^p}^p} \]
where the supremum is taken over all nonzero functions \( f \in h^p(H) \); the constants suppressed in this estimate depend only on \( p \). For measures supported on \( H \), one may find a proof of these statements in [9, Theorem I.5.6] and its proof. Extension to the measures on \( \overline{H} \) is standard, as in the case of the ball (see [4, Theorem 2.35]).

Furthermore, when \( \mu \) is an \( h^p \)-Carleson measure, its compactness is characterized as follows:

- \( \mu \) is a compact \( h^p \)-Carleson measure \( \iff M_\delta(\mu) \to 0 \) as \( \delta \downarrow 0 \).

To see this one may easily modify the proof of [9, Theorem I.5.6]. As a consequence, the notions of (compact) \( L^p(\sigma) \)-Carleson measures do not even depend on \( p \).

Finally, for the Hardy spaces, we remark that the (compact) \( H^p \)-Carleson measure characterizations above extend to the range \( 0 < p \leq 1 \); see [9, Theorem I.3.9].

2.5. Pullback measures. For \( \varphi \in S(H) \), recall that \( \mu_{\varphi} \) denotes the pullback measure \( m \circ \varphi^{-1} \) on \( \overline{H} \). Use of a change-of-variable formula from measure theory ([10, p.163]) shows that
\[ \int_R h \circ \varphi(x) \, dx = \int_{\overline{H}} h(w) \, d\mu_{\varphi}(w) \]
for positive Borel functions \( h \) on \( \overline{H} \). For example, given \( \sigma \in \{ b, h \} \) and \( 1 \leq p < \infty \), we have
\[ \| \mathcal{X}^\sigma(z, \varphi(\cdot)) \|^p = \int_{\overline{H}} |\mathcal{X}^\sigma(z, w)|^p \, d\mu_{\varphi}(w), \quad z \in \overline{H} \]
and
\[ \| S_\sigma^\varphi f \|^p = \int_{\overline{H}} |f^\sigma(w)|^p \, d\mu_{\varphi}(w) \]
for functions \( f \in L^p(R) \).

When \( 1 < p < \infty \), note from (2.11) that \( S_\varphi^\sigma \) is bounded (respectively compact) on \( L^p(R) \) if and only if \( \mu_{\varphi} \) is a (compact) \( L^p(\sigma) \)-Carleson measure.

2.6. Composition operators. We now show that Sarason’s composition operators induced by the boundary functions of holomorphic self-maps of \( H \), acting on Hardy spaces, reduce to ordinary composition operators. Given a complex function \( g \) on \( H \) and \( z \in \overline{H} \), we will use the notation
\[ g^*(z) = \lim_{y \downarrow 0} g(z + iy) \]
whenever this limit exists. Also, we denote by \( S_\alpha(H) \) the class of all holomorphic self-maps of \( H \). Given \( \Phi \in S_\alpha(H) \), we denote by \( C_\Phi \) the composition operator given by
\[ C_\Phi g = g \circ \Phi \]
for holomorphic functions $g$ on $H$.

For $\Phi \in S_a(H)$, we first note $\Phi$ satisfies (1.1), and so $\phi = \Phi^*$ and $\Phi^* \in S(H)$. The corresponding statement for self-maps of the unit disk is [15, Lemma 2], and this can be easily transferred to our setting of self-maps of $H$. So, identifying $\Phi$ with $\Phi^*$, we see that $S_a(H)$ is naturally embedded into $S(H)$.

The disk analogue of the next proposition is well known; see [4, Proposition 2.25].

**Proposition 2.1.** Let $0 < p \leq \infty$ and $\Phi \in S_a(H)$. Put $\phi = \Phi^*$. Then $g^* \circ \phi = (g \circ \Phi)^*$ m-a.e. on $R$ for $g \in H^p(H)$.

**Proof.** Let $g \in H^p(H)$. Fix a Riemann map $\tau$ of the unit disk $D$ onto $H$. Since $g$ has a harmonic majorant by [7, Theorem 11.1], $g \circ \tau$ is an $H^p$-function on $D$. It follows from the well-known theory of Nevanlinna class functions that $g \circ \tau$ is a quotient of bounded holomorphic functions on $D$. Thus $g$ is a quotient of bounded holomorphic functions on $H$. Now, one may imitate the proof of [4, Proposition 2.25] over $D$, depending on the Lindelöf Theorem, to conclude the proposition. $\square$

Now, for $\phi = \Phi^*$ with $\Phi \in S_a(H)$, we have $(S^*_\sigma g^*) = (C_\Phi g)^*$ for $\sigma \in \{b, h\}$ and functions $g \in H^p(H)$, $1 \leq p \leq \infty$, by Proposition 2.1. This shows that Sarason’s composition operators $S^b_\sigma$ and $S^h_\sigma$, when acting on the Hardy spaces $H^p(R)$, are equal and can be identified with the ordinary composition operator $C_\Phi$.

### 2.7. Miscellany.

For later reference, we mention an elementary result (see, for example, [13, p. 90] or [11, Lemma 3.17]) from real analysis that will be used repeatedly, following the approaches of [3] and [15].

**Lemma 2.2.** Let $\nu$ be a positive measure on a set $\Omega$. Let $g \in L^1(\nu)$ and $\{g_n\}$ be a sequence in $L^1(\nu)$ such that $g_n \to g$ $\nu$-a.e. on $\Omega$. Then $\|g_n\|_{L^1(\nu)} \to \|g\|_{L^1(\nu)}$ if and only if $g_n \to g$ in $L^1(\nu)$.

Finally, we set some notation. We put

$$\|A\|_H := \sup_{z \in H} A(z)$$

for a nonnegative function $A$ on $H$. Also, given a bounded linear operator $S$ taking a Banach space $X$ into another Banach space $Y$, we denote by

$$\|S\|_{X \to Y}$$

the operator norm of $S$. We put

$$\|S\|_X := \|S\|_{X \to X}$$

for simplicity.

### 3. $L^1$-Characterizations

In this section we prove a strengthened version of Theorem 1.1 including boundedness/compactness on $M(R)$, observe some consequences and provide some examples.

Before proceeding, we set some notation relevant to $M(R)$. Given $\tau \in M(R)$ and $\sigma \in \{b, h\}$, we denote by $\tau^\sigma$ the (harmonic or holomorphic) function on $H$ defined by

$$\tau^\sigma(z) := \int_{-\infty}^{\infty} K^\sigma(z, t) \, d\tau(t)$$
for \( z \in H \). As in the case of functions, it is well known that \( \tau^b \), and hence \( \tau^b \) as well, has vertical limits \( m \text{-a.e.} \) on \( R \); see [6, Proposition 10.4.1]. So, we may extend \( \tau^\sigma \) to \( \tilde{H} \setminus E \), where \( E \subset R \) is an \( m \)-null set, by defining
\[
(3.3) \quad \tau^\sigma(z) = \lim_{y \downarrow 0} \tau^\sigma(z + iy)
\]
for \( z \in \tilde{H} \setminus E \). Now, for \( \varphi \in S(H) \), we define on \( R \setminus \varphi^{-1}(E) \)
\[
S^\sigma_\varphi \tau := \tau^\sigma \circ \varphi.
\]
Notice that \( \varphi^{-1}(E) \) is an \( m \)-null set by (1.1), so \( S^\sigma_\varphi \tau \) has been defined \( m \text{-a.e.} \). This definition, of course, agrees with the earlier one when \( d\tau = f \, dm \) for \( f \in L^1(R) \).

We first prove the boundedness part. In the course of the proof we also obtain the precise operator norms, as in the next theorem.

**Theorem 3.1.** Let \( \varphi \in S(H) \) and \( \sigma \in \{b, h\} \). Then the following statements are equivalent:

1. \( S^\sigma_\varphi : M(R) \to L^1(R) \) is bounded;
2. \( S^\sigma_\varphi \) is bounded on \( L^1(R) \);
3. \( \Lambda^\sigma_{\varphi,1} \) is bounded on \( H \).

Moreover, the operator norms satisfy
\[
\|S^\sigma_\varphi\|_{M(R) \to L^1(R)} = \|S^\sigma_\varphi\|_{L^1(R)} = \|\Lambda^\sigma_{\varphi,1}\|_{H}.
\]

**Proof.** The implication (a) \( \implies \) (b) is clear. We now prove the implication (b) \( \implies \) (c). Assume that \( S^\sigma_\varphi \) is bounded on \( L^1(R) \). Let \( z \in H \). Choosing \( P_z = K^h(z, \cdot) \) as a test function, we have by (2.4)
\[
(3.2) \quad S^\sigma_\varphi P_z = \mathcal{K}^\sigma(z, \varphi(\cdot)).
\]
Now, since \( \|P_z\|_1 = 1 \), integration on \( R \) yields
\[
\|S^\sigma_\varphi\|_{L^1(R)} \geq \|S^\sigma_\varphi P_z\|_1 = \|\mathcal{K}^\sigma(z, \varphi(\cdot))\|_1 = \Lambda^\sigma_{\varphi,1}(z).
\]
This being true for arbitrary \( z \in H \), we conclude
\[
(3.3) \quad \|S^\sigma_\varphi\|_{L^1(R)} \geq \|\Lambda^\sigma_{\varphi,1}\|_{H},
\]
as required.

Finally, we prove the implication (c) \( \implies \) (a). Assume that \( \Lambda^\sigma_{\varphi,1} \) is bounded on \( H \). Let \( \tau \in M(R) \). Note from (3.1), (2.1) and (1.1)
\[
(3.4) \quad (\tau^\sigma \circ \varphi)(x) = \lim_{y \downarrow 0} \int_{-\infty}^{\infty} \mathcal{K}^\sigma(x, t + iy) \, d\tau(t)
\]
at \( m \)-almost every \( x \in R \). So, we have by Fatou’s Lemma and Fubini’s Theorem
\[
\|S^\sigma_\varphi \tau\|_1 = \int_{H} |\tau^\sigma(z)| \, d\mu^\varphi(z) \quad \text{(by (2.11))}
\]
\[
\leq \liminf_{y \downarrow 0} \int_{-\infty}^{\infty} \left\{ \int_{H} |\mathcal{K}^\sigma(z, t + iy)| \, d\mu^\varphi(z) \right\} \, d|\tau|(t)
\]
\[
= \liminf_{y \downarrow 0} \int_{-\infty}^{\infty} \Lambda^\sigma_{\varphi,1}(t + iy) \, d|\tau|(t) \quad \text{(by (2.10))}
\]
\[
\leq \|\Lambda^\sigma_{\varphi,1}\|_{H} \|\tau\|_{M}.
\]
Since this holds for every \( \tau \in M(\mathbb{R}) \), we conclude that \( S^\sigma_\varphi : M(\mathbb{R}) \to L^1(\mathbb{R}) \) is bounded with norm estimate
\[
\|S^\sigma_\varphi\|_{M(\mathbb{R}) \to L^1(\mathbb{R})} \leq \|A^\sigma_{\varphi,1}\|_H.
\]
(3.5)

This completes the proof of equivalences of (a), (b) and (c).

By (3.1), we obtain the next corollary.

\[\square\]

This completes the proof of equivalences of (a), (b) and (c).

For \( \varphi \in S(\mathbb{H}) \) and \( \varepsilon > 0 \), note that \( \varphi + \varepsilon i \in S(\mathbb{H}) \), because \( \mu_{\varphi + \varepsilon i}|_{\mathbb{R}} \) is the zero measure. As a consequence of Theorem 3.1, we obtain the next corollary.

**Corollary 3.2.** Let \( \varphi \in S(\mathbb{H}) \) and \( \sigma \in \{b, h\} \). If \( S^\sigma_\varphi \) is bounded on \( L^1(\mathbb{R}) \), then so is \( S^\sigma_{\varphi + \varepsilon i} \) for each \( \varepsilon > 0 \). Moreover, \( \|S^\sigma_{\varphi + \varepsilon i}\|_{L^1(\mathbb{R})} \) increases to \( \|S^\sigma_\varphi\|_{L^1(\mathbb{R})} \) as \( \varepsilon \downarrow 0 \).

**Proof.** Note from (2.1)
\[
(3.6) \quad \Lambda^\sigma_{\varphi,1}(z + \varepsilon i) = \Lambda^\sigma_{\varphi + \varepsilon i,1}(z)
\]
for all \( z \in \mathbb{H} \). Thus \( \|\Lambda^\sigma_{\varphi + \varepsilon i,1}\|_H \) is increasing as \( \varepsilon \downarrow 0 \) and
\[
(3.7) \quad \lim_{\varepsilon \downarrow 0} \|\Lambda^\sigma_{\varphi + \varepsilon i,1}\|_H \leq \|\Lambda^\sigma_{\varphi,1}\|_H.
\]
Meanwhile, we have by (3.6) and Fatou’s Lemma
\[
\Lambda^\sigma_{\varphi,1}(z) \leq \liminf_{\varepsilon \downarrow 0} \Lambda^\sigma_{\varphi + \varepsilon i,1}(z)
\]
for all \( z \in \mathbb{H} \). This implies that the inequality in (3.7) can be reversed. So, the corollary holds by Theorem 3.1. \[\square\]

Note from (2.2) that \( \Lambda^b_{\varphi,1} \leq 2\Lambda^b_{\varphi,1} \) on \( \mathbb{H} \). So, we also have the following corollary of Theorem 3.1.

**Corollary 3.3.** Let \( \varphi \in S(\mathbb{H}) \). If \( S^b_\varphi \) is bounded on \( L^1(\mathbb{R}) \), then so is \( S^b_{\varphi} \). Moreover, \( \|S^b_\varphi\|_{L^1(\mathbb{R})} \) decreases to \( \|S^b_{\varphi}\|_{L^1(\mathbb{R})} \).

Certainly, the converse of the above corollary fails. For example, take \( \varphi = id \), the identity function on \( \mathbb{R} \). Note from (1.3) that \( S^b_{id} \) is the identity operator on \( L^1(\mathbb{R}) \). On the other hand, \( S^b_{id} \) is the Borel transform followed by the vertical limiting process, which is not bounded on \( L^1(\mathbb{R}) \), as was mentioned in Section 2.3. In fact we have the following necessary condition for \( L^1 \)-boundedness of \( S^b_\varphi \).

**Corollary 3.4.** Let \( \varphi \in S(\mathbb{H}) \). If \( S^b_\varphi \) is bounded on \( L^1(\mathbb{R}) \), then \( \mu_\varphi(\mathbb{R}) = 0 \).

**Proof.** It is easily seen from Fatou’s Lemma that
\[
\Lambda^b_{\varphi,1}(x) \leq \|\Lambda^b_{\varphi,1}\|_H
\]
for all \( x \in \mathbb{R} \). Thus, for any \( R > 0 \), we have
\[
2R\|\Lambda^b_{\varphi,1}\|_H \geq \int_{-R}^{R} \Lambda^b_{\varphi,1}(x) \, dx
\]
\[
= \frac{1}{2\pi} \int_{-R}^{R} \int_{-\infty}^{\infty} \frac{dt}{|\varphi(x) - t|} \, dx
\]
\[
\geq \frac{1}{2\pi} \int_{E(R)} \int_{-\infty}^{\infty} \frac{dt}{|\varphi(x) - t|} \, dx,
\]
where \( E(R) = [-R, R] \cap \varphi^{-1}(R) \). Noting that the inner integral of the above diverges for each \( x \in E(R) \), when \( S_1^\varphi \) is bounded on \( L^1(R) \) it follows from Theorem 3.1 that \( m|E(R)| = 0 \). Since this holds for all \( R > 0 \), the proof is complete. \( \square \)

**Corollary 3.5.** Let \( \varphi \in S(H) \) and \( \sigma \in \{ b, h \} \). If \( \varphi(R) \) is contained in a compact subset of \( H \), then \( S_\varphi^\sigma \) is not bounded on \( L^1(R) \).

*Proof.* Suppose that \( \varphi(R) \) is contained in a compact subset of \( H \). Choose positive numbers \( \epsilon \) and \( R \) such that \( |\varphi(x)| \leq R \) and \( \text{Im} \varphi(x) \geq \epsilon \) for all \( x \in R \). Now, since

\[
\mathcal{X}^h(\varphi(x), z) = \frac{\text{Im} \varphi(x) + \text{Im} z}{|\varphi(x) - \overline{z}|^2} \geq \frac{\epsilon + \text{Im} z}{(R + |z|)^2}
\]

for all \( x \in R \) and \( z \in H \), we see that \( \Lambda_{\varphi,1}^h \) is identically \( \infty \) on \( H \). Thus the corollary holds by Theorem 3.1 and Corollary 3.3. \( \square \)

We now proceed to the investigation of when the operators under consideration are compact on \( L^1(R) \).

**Lemma 3.6.** Let \( \varphi \in S(H) \) and assume \( \Lambda_{\varphi,1}^h(z_0) < \infty \) at some \( z_0 \in H \). Then \( \mu_\varphi(K) < \infty \) for any compact set \( K \subset \overline{H} \).

*Proof.* Setting

\[
E_r(z) := \{ w \in \overline{H} : |w - \overline{z}| < r\text{Im} \ z \}
\]

for \( r > 1 \) and \( z \in H \), we have

\[
\frac{\mu_\varphi[E_r(z)]}{\text{Im} z} = \int_{E_r(z)} \frac{d\mu_\varphi(w)}{\text{Im} z} \leq \int_{E_r(z)} \frac{\text{Im} z + \text{Im} w}{(\text{Im} z)^2} d\mu_\varphi(w) \leq \pi r^2 \int_{E_r(z)} \mathcal{X}^h(z, w) d\mu_\varphi(w).
\]

This yields

\[
\frac{\mu_\varphi[E_r(z)]}{\text{Im} z} \leq \pi r^2 \Lambda_{\varphi,1}^h(z)
\]

for all \( r > 1 \) and \( z \in H \). This implies the lemma. \( \square \)

**Lemma 3.7.** Let \( \varphi \in S(H) \) and \( \sigma \in \{ b, h \} \). Assume \( \varphi(R) \subset H + \epsilon i \) for some \( \epsilon > 0 \). Also, assume \( \Lambda_{\varphi,1}^\sigma \mid_R \subset L^\infty(R) \) and \( \Lambda_{\varphi,1}^\sigma \mid_R \) is continuous at \( \infty \). Then \( S_\varphi^\sigma : M(R) \to L^1(R) \) is compact.

*Proof.* Let \( \{ \tau_k \} \) be a sequence in \( M(R) \) with \( \| \tau_k \|_M \leq 1 \) for all \( n \). To show \( S_\varphi^\sigma \) is compact, we need to show that there is an \( L^1 \)-norm convergent subsequence of \( \{ S_\varphi^\sigma \tau_k \} \). By passing to a subsequence if necessary we may assume that \( \tau_k \) weak-star converges (in the topology of the dual of \( C_0(R) \)) to some \( \tau \in M(R) \) with \( \| \tau \|_M \leq 1 \). Note \( K_{\varphi}^\sigma(z, \cdot) \in C_0(R) \) for each \( z \in H \). Thus we see from the weak-star convergence that \( \tau_k \to \tau \) pointwise on \( H \). Also, note from the assumption \( \varphi(R) \subset H + \epsilon i \) that
$S^\sigma \tau_k(x) = \tau^\sigma_\varphi(x)$ and $S^\sigma \tau(x) = \tau^\sigma_\varphi(x)$ for all $x \in \mathbb{R}$. It follows from Fatou's Lemma and Fubini's Theorem that
\[
\int_{-\infty}^{\infty} |\tau^\sigma \circ \varphi| \, dm = \int_{-\infty}^{\infty} \left| \lim_{k \to \infty} \int_{-\infty}^{\infty} K^\sigma (\varphi(x), t) \, d\tau_k(t) \right| \, dx \\
\leq \liminf_{k \to \infty} \int_{-\infty}^{\infty} \Lambda^\sigma_{\varphi,1}(t) \, d\tau_k(t) \\
\leq \liminf_{k \to \infty} \|\tau_k\|_M \|\Lambda^\sigma_{\varphi,1}\|_R \\
\leq \|\Lambda^\sigma_{\varphi,1}\|_R.
\]
In particular, we have $\tau^\sigma \circ \varphi \in L^1(\mathbb{R})$ by the assumption that $\Lambda^\sigma_{\varphi,1}|_R \in L^\infty(\mathbb{R})$.

Now, in order to complete the proof, it suffices to show
\[
(3.8) \quad \|S^\sigma_\varphi(\tau_k - \tau)\|_1 \to 0
\]
as $k \to \infty$.

We now proceed to the proof of (3.8). First, we note
\[
\|K^\sigma(z, \cdot)\|_\infty \leq \frac{1}{\pi \mathrm{Im} \, z} \quad \text{on} \quad \mathbb{R}
\]
so that
\[
\tau_k^\sigma(z) \leq \|\tau_k\|_M \frac{M}{\pi \mathrm{Im} \, z} \leq \frac{1}{\pi \mathrm{Im} \, z}
\]
for all $k$ and $z \in \mathbb{H}$. This implies that $\{\tau_k^\sigma\}$ is a normal family of harmonic or holomorphic functions. Thus, by passing to another subsequence if necessary, we see that $\tau_k^\sigma \to \tau^\sigma$ uniformly on compact subsets of $\mathbb{H}$. Now, pick $t_\sigma \in \mathbb{R}$ such that $\Lambda^\sigma_{\varphi,1}(t_\sigma) < \infty$ and fix $r > |t_\sigma|$. We split the integral in question into two pieces as follows:
\[
\| (\tau_k^\sigma - \tau^\sigma) \circ \varphi \|_1 = \int_{\mathbb{H} + ei} |\tau_k^\sigma - \tau^\sigma| \, d\mu_\varphi \\
= \int_{D_r(0) \cap (\mathbb{H} + ei)} + \int_{(\mathbb{H} + ei) \setminus D_r(0)}
\]
where $D_r(0)$ is the set specified in (2.8). From the assumption that $\varphi(\mathbb{R}) \subset \mathbb{H} + ei$, we have $\varphi - ei \in S(\mathbb{H})$ and thus $\Lambda^\sigma_{\varphi - ei,1}(t_\sigma + ei) = \Lambda^\sigma_{\varphi,1}(t_\sigma) < \infty$ by (3.6). Hence $\mu_{\varphi - ei}(K) < \infty$ for any compact $K \subset \mathbb{H}$ by Lemma 3.6. Since $\mu_{\varphi - ei}(K) = \mu_\varphi(K + ei)$, it follows that $\mu_\varphi[D_r(0) \cap (\mathbb{H} + ei)] < \infty$. Thus the first integral above tends to 0 as $k \to \infty$ by the uniform convergence $\tau_k^\sigma \to \tau^\sigma$ on $D_r(0) \cap (\mathbb{H} + ei)$. Accordingly, setting
\[
J_k(r) := \int_{(\mathbb{H} + ei) \setminus D_r(0)} |\tau_k^\sigma - \tau^\sigma| \, d\mu_\varphi,
\]
we obtain
\[
(3.9) \quad \limsup_{k \to \infty} \| (\tau_k^\sigma - \tau^\sigma) \circ \varphi \|_1 \leq \limsup_{k \to \infty} J_k(r)
\]
for each fixed $r > |t_0|$. Now, setting
\[
\nu_k := \tau_k - \tau,
\]
we see by Fubini’s Theorem that
\[ J_k(r) \leq \int_{-\infty}^{\infty} \int_{|z| \geq r} |K^\sigma(z,t)| \, d\mu_\varphi(z) \, d|\nu_k|(t) \]
(3.10)
\[ \leq \int_{|t| < r/2} \int_{|z| \geq r} + \int_{|t| \geq r/2} \int_{\mathfrak{T}} \]
\[ =: I_k(r) + II_k(r). \]

For the second term of the above, we note
\[ \|\nu_k\|_M \leq \|\tau_k\|_M + \|\tau\|_M \leq 2 \]
(3.11)
for all \( k \) and thus
\[ \sup_k II_k(r) \leq 2 \sup_{|t| \geq r/2} \Lambda_{\varphi,1}(t). \]
(3.12)

For the first term, for \( |z| \geq r \) and \( |t| < r/2 \), we note
\[ 4|z-t| \geq 2|z| \geq |z| + r > |z-t_\sigma| \]
and hence
\[ |K^b(z,t)| \leq 4|K^b(z,t_\sigma)| \quad \text{and} \quad |K^h(z,t)| \leq 16|K^h(z,t_\sigma)|. \]

So, we obtain by Fubini’s Theorem and (3.11)
\[ \sup_k I_k(r) \leq 32 \int_{|z| \geq r} |K^\sigma(z,t_\sigma)| \, d\mu_\varphi(z). \]

Combining this with (3.9), (3.10) and (3.12), we obtain
\[ \limsup_{k \to \infty} \|(\tau_k^\sigma - \tau^\sigma) \circ \varphi\|_1 \leq 2 \sup_{|t| \geq r/2} \Lambda_{\varphi,1}(t) + 32 \int_{|z| \geq r} |K^\sigma(z,t_\sigma)| \, d\mu_\varphi(z) \]
(3.13)
for each \( r > |t_0| \). Note that the first term in the right hand side of the above tends to 0 as \( r \to \infty \) by continuity of \( \Lambda_{\varphi,1}(t_\sigma)^{\mathfrak{T}} \) at \( \infty \). Meanwhile, since \( \Lambda_{\varphi,1}(t_\sigma) < \infty \) by choice of \( t_\sigma \), the second term also tends to 0 as \( r \to \infty \) by the Dominated Convergence Theorem. Thus, taking the limit \( r \to \infty \) in the right hand side of (3.13), we conclude the lemma. The proof is complete. \( \square \)

We need some additional notation. Let \( \sigma \in \{b, h\} \). For a function \( \varphi \in \mathcal{S}(\mathfrak{H}) \), we put
\[ \omega_\sigma^\varphi(R) := \sup_{|z| \geq R, z \in \mathfrak{T}} \Lambda_{\varphi,1}(z) \]
for \( R > 0 \). We note for easier reference later that
\[ \lim_{R \to \infty} \omega_\sigma^\varphi(R) = 0 \iff \Lambda_{\varphi,1}^\sigma \text{ : continuous at } \infty; \]
recall \( \Lambda_{\varphi,1}^\sigma(\infty) = 0 \) by definition. Also, we define
\[ Q_{\varphi}^\sigma(t + si) := \int_{\mathfrak{T}} |\mathcal{K}^\sigma(z, t + si) - K^\sigma(z, t)| \, d\mu_\varphi(z) \]
for \( t \in \mathfrak{R} \) and \( s \geq 0 \). Note \( Q_{\varphi}^\sigma = 0 \) on \( \mathfrak{R} \). Finally, we put
\[ \Sigma_c := \{z \in \mathfrak{H} : 0 \leq \text{Im} \, z \leq c\} \]
for \( c > 0 \).

**Lemma 3.8.** Let \( \varphi \in \mathcal{S}(\mathfrak{H}) \) and \( \sigma \in \{b, h\} \). If \( \Lambda_{\varphi,1}^\sigma \in C(\mathfrak{H}) \), then \( Q_{\varphi}^\sigma \) is uniformly continuous on \( \Sigma_c \) for each \( c > 0 \).
Proof. Assume $\Lambda_{\varphi,1}^\sigma \in C(\hat{\mathbf{H}})$. Let $z, w \in \hat{\mathbf{H}}$. Note
\begin{equation}
|Q_{\varphi}^\sigma(z) - Q_{\varphi}^\sigma(w)| \leq \int_{\overline{\Pi}} |\mathcal{K}^\sigma(\xi, z) - \mathcal{K}^\sigma(\xi, w)| \, d\mu_\varphi(\xi).
\end{equation}
Also, note
\[\lim_{z \to w} \int_{\overline{\Pi}} |\mathcal{K}^\sigma(\xi, z)| \, d\mu_\varphi(\xi) = \int_{\overline{\Pi}} |\mathcal{K}^\sigma(\xi, w)| \, d\mu_\varphi(\xi),\]
from the continuity of $\Lambda_{\varphi,1}^\sigma$ at $w$. So, we have by Lemma 2.2
\[\lim_{z \to w} \int_{\overline{\Pi}} |\mathcal{K}^\sigma(\xi, z) - \mathcal{K}^\sigma(\xi, w)| \, d\mu_\varphi(\xi) = 0.\]
Thus, taking the limit $z \to w$ in (3.16), we see that
\begin{equation}
Q_{\varphi}^\sigma \text{ is continuous on } \hat{\mathbf{H}}.
\end{equation}
Meanwhile, since
\[Q_{\varphi}^\sigma(t + si) \leq \Lambda_{\varphi,1}^\sigma(t + si) + \Lambda_{\varphi,1}^\sigma(t) \leq 2\omega_\sigma(|t|)\]
for $t \in \mathbf{R}$ and $s \geq 0$, we see from (3.14) that
\begin{equation}
Q_{\varphi}^\sigma(t + si) \to 0 \text{ uniformly in } s \geq 0
\end{equation}
as $|t| \to \infty$. Now, given $c > 0$, we see from (3.17) and (3.18) that the function $Q_{\varphi}^\sigma$, when restricted to the strip $\Sigma_c$, continuously extends to $\infty$. In other words, the restricted function $Q_{\varphi}^\sigma|_{\Sigma_c}$ continuously extends to $\Sigma_c \cup \{\infty\}$. Now, since $\Sigma_c \cup \{\infty\}$ is a compact subset of $\hat{\mathbf{H}}$, we conclude the lemma. The proof is complete. \hfill \Box

We are now ready to prove the compactness part of Theorem 1.1.

**Theorem 3.9.** Let $\varphi \in \mathcal{S}(\mathbf{H})$ and $\sigma \in \{b, h\}$. Then the following statements are equivalent:

(a) $S_{\varphi}^\sigma : M(\mathbf{R}) \to L^1(\mathbf{R})$ is compact;
(b) $S_{\varphi}^\sigma$ is compact on $L^1(\mathbf{R})$;
(c) $\Lambda_{\varphi,1}^\sigma \in C(\hat{\mathbf{H}})$.

**Proof.** The implication (a) $\implies$ (b) is clear. We now prove the implication (b) $\implies$ (c); we remark that the proof of this implication for the ball version ([3, Proposition 4.6]) contains a minor error and that the proof here is a corrected version for the half-plane. Assume that $S_{\varphi}^\sigma$ is compact on $L^1(\mathbf{R})$. Note from Theorem 3.1 that $\Lambda_{\varphi,1}^\sigma$ is bounded on $\mathbf{H}$, and thus on $\hat{\mathbf{H}}$ by Fatou's Lemma. To prove the continuity of $\Lambda_{\varphi,1}^\sigma$ on $\hat{\mathbf{H}}$, it suffices to show
\begin{equation}
\lim_{z \to w, z \in \hat{\mathbf{H}}} \Lambda_{\varphi,1}^\sigma(z) = \Lambda_{\varphi,1}^\sigma(w)
\end{equation}
for each $w \in \hat{\mathbf{H}}$.

Fix $w \in \hat{\mathbf{H}}$ and let $\{z_j\}$ be a sequence of points in $\mathbf{H}$ such that $z_j \to w$. Since $\|P_{\varphi}\|_1 = 1$ for all $j$ and $S_{\varphi}^\sigma$ is compact on $L^1(\mathbf{R})$, it follows from (3.2) that the sequence $\{S_{\varphi}^\sigma P_{\varphi}\} = \{\mathcal{K}^\sigma(z_j, \varphi(\cdot))\}$ has a subsequence $\{\mathcal{K}^\sigma(z_{j_k}, \varphi(\cdot))\}$ that converges in norm. Since $\mathcal{K}^\sigma(z_j, \varphi(\cdot)) \to \mathcal{K}^\sigma(w, \varphi(\cdot))$ pointwise $m$-a.e. on $\mathbf{R}$, it follows that $\mathcal{K}^\sigma(z_{j_k}, \varphi(\cdot)) \to \mathcal{K}^\sigma(w, \varphi(\cdot))$ in norm. We thus obtain
\[\Lambda_{\varphi,1}^\sigma(z_{j_k}) = \|\mathcal{K}^\sigma(z_{j_k}, \varphi(\cdot))\|_1 \to \|\mathcal{K}^\sigma(w, \varphi(\cdot))\|_1 = \Lambda_{\varphi,1}^\sigma(w).\]
This subsequential limit property implies (3.19) and thus completes the proof that \( \Lambda_{\varphi,1}^\sigma \in C(\hat{H}) \).

Finally, we prove the implication (c) \( \implies \) (a). Assume \( \Lambda_{\varphi,1}^\sigma \in C(\hat{H}) \). Note from Theorem 3.1 that \( S_{\varphi}^\sigma : M(R) \to L^1(R) \) is bounded. Also, note from Lemma 3.7 that \( S_{\varphi+\epsilon}^\sigma : M(R) \to L^1(R) \) is compact for each \( \epsilon > 0 \). So, it suffices to show
\[
\|S_{\varphi}^\sigma - S_{\varphi+\epsilon}^\sigma\|_{M(R) \to L^1(R)} \to 0
\]
as \( \epsilon \downarrow 0 \).

Note that \( Q_{\varphi}^\sigma \) is uniformly continuous on the strip \( \Sigma_1 \) by Lemma 3.8. Thus we have
\[
Q_{\varphi}^\sigma(x + yi) \to 0 \quad \text{uniformly in} \quad x \in R
\]
as \( y \downarrow 0 \). Meanwhile, given \( \tau \in M(R) \), we have by (3.4), Fatou’s Lemma and Fubini’s Theorem
\[
\|(S_{\varphi}^\sigma - S_{\varphi+\epsilon}^\sigma)\tau\|_1 \\
\leq \liminf_{y \downarrow 0} \int_{-\infty}^{\infty} \left\{ \int_{\mathcal{H}} |\mathcal{X}^\sigma(z, x + yi) - \mathcal{X}^\sigma(z, x + yi + \epsilon)| \, d\mu_{\varphi}(z) \right\} \, d|\tau|(x) \\
\leq \liminf_{y \downarrow 0} \left( \sup_{x \in R} \left[ Q_{\varphi}^\sigma(x + yi) + Q_{\varphi}^\sigma(x + yi + \epsilon) \right] \right) \|\tau\|_M.
\]
Now, since \( \tau \in M(R) \) is arbitrary, this, together with (3.21), yields
\[
\|S_{\varphi}^\sigma - S_{\varphi+\epsilon}^\sigma\|_{M(R) \to L^1(R)} \leq \liminf_{y \downarrow 0} \left[ \sup_{x \in R} Q_{\varphi}^\sigma(x + yi + \epsilon) \right] \\
\leq \sup_{0 < \epsilon \leq 2\epsilon} \left[ \sup_{x \in R} Q_{\varphi}^\sigma(x + si) \right]
\]
for each \( \epsilon > 0 \). Now, taking the limit \( \epsilon \downarrow 0 \) and using (3.21), we conclude (3.20), as asserted. This completes the proof. \( \square \)

We now observe a couple of consequences of Theorem 3.9.

**Corollary 3.10.** Let \( \varphi \in S(H) \). If \( S_{\varphi}^b \) is compact on \( L^1(R) \), then so is \( S_{\varphi}^h \).

**Proof.** Note from (2.2)
\[
\|\mathcal{X}^b(z, \varphi(\cdot)) - \mathcal{X}^b(w, \varphi(\cdot))\|_1 \leq 2 \|\mathcal{X}^b(z, \varphi(\cdot)) - \mathcal{X}^b(w, \varphi(\cdot))\|_1
\]
for \( z, w \in \mathcal{H} \). We thus see by Lemma 2.2 that \( \Lambda_{\varphi,1}^b \in C(\hat{H}) \) implies \( \Lambda_{\varphi,1}^b \in C(\hat{H}) \), and so conclude the corollary by Theorem 3.9. \( \square \)

The next corollary shows a class of symbol functions that induce compact operators.

**Corollary 3.11.** Let \( \varphi \in S(H) \). If \( \frac{1}{|\text{Im} \varphi|} \in L^1(R) \), then \( S_{\varphi}^b \) is compact on \( L^1(R) \).

**Proof.** Note that
\[
|\mathcal{X}^b(z, \varphi(\cdot))| \leq \frac{1}{2\pi} \cdot \frac{1}{|\text{Im} \varphi(\cdot)|} \quad \text{on} \quad R
\]
for all \( z \in \mathcal{H} \). This, together with the Dominated Convergence Theorem, implies that \( \Lambda_{\varphi,1}^b \) is continuous on \( \mathcal{H} \). So, the corollary holds by Theorem 3.9. \( \square \)
The sufficient condition \( \frac{1}{|x|^\alpha} \in L^1(\mathbb{R}) \) in Corollary 3.11 is far from being necessary. Here, we provide a class of explicit examples.

**Example 3.12.** For \( \alpha > 0 \), let \( \varphi_\alpha \in \mathcal{S}(\mathbb{H}) \) be the function defined by

\[
\varphi_\alpha(x) := |x|^\alpha + i, \quad x \in \mathbb{R}.
\]

Then the following statements hold:

(a) \( S^b_{\varphi_\alpha} \) is bounded/compact on \( L^1(\mathbb{R}) \) if and only if \( \alpha > 1 \);

(b) \( S^h_{\varphi_\alpha} \) is bounded on \( L^1(\mathbb{R}) \) if and only if \( \alpha \geq 1 \);

(c) \( S^h_{\varphi_\alpha} \) is compact on \( L^1(\mathbb{R}) \) if and only if \( \alpha > 1 \).

**Proof.** Since our proof of (a) is rather long and complicated, we first prove (b) and (c). For \( 0 < \alpha \leq \frac{1}{2} \), note \( \Lambda^b_{\varphi_\alpha, 1} = \infty \) on \( \mathbb{H} \), because \( |x|^{-2\alpha} \) is not integrable near \( \infty \). Thus \( S^h_{\varphi_\alpha} \) is not bounded on \( L^1(\mathbb{R}) \) by Theorem 3.1.

For \( \frac{1}{2} < \alpha < 1 \), note for \( x \in \mathbb{R} \) and \( y > 0 \),

\[
\Lambda^b_{\varphi_\alpha, 1}(x + iy) = \frac{2}{\pi} \int_{-x/(y+1)}^{\infty} \frac{ds}{s^{\frac{1}{\alpha} - 1} + 1} \leq 2
\]

for all \( x \in \mathbb{R} \) and \( y > 0 \). It follows that \( S^h_{\varphi_\alpha} \) is bounded on \( L^1(\mathbb{R}) \) by Theorem 3.1, but not compact by Theorem 3.9.

For \( \alpha > 1 \), \( S^h_{\varphi_\alpha} \) is compact on \( L^1(\mathbb{R}) \) by (a) (to be proved below) and Corollary 3.10. This completes the proof of (b) and (c).

We turn to the proof of (a). First we consider \( 0 < \alpha \leq 1 \). Then \( |x|^{-\alpha} \) is not integrable near \( \infty \), and hence \( S^b_{\varphi_\alpha} \) is not bounded on \( L^1(\mathbb{R}) \) as in the proof of (b) and (c). We now assume \( \alpha > 1 \) and show that \( S^b_{\varphi_\alpha} \) is compact on \( L^1(\mathbb{R}) \). It suffices to show \( \Lambda^b_{\varphi_\alpha, 1} \in C(\mathbb{H}) \) by Theorem 3.9. Note

\[
\Lambda^b_{\varphi_\alpha, 1}(x + iy) = \frac{1}{\pi} \int_{0}^{\infty} \frac{dt}{|t^\alpha - x + i(y+1)|}
\]

for \( x \in \mathbb{R} \) and \( y \geq 0 \). Since

\[
\frac{1}{|t^\alpha - x + i(y+1)|} \leq \frac{1}{|t^\alpha - x + i|}
\]

for all \( t \in \mathbb{R} \) and \( y \geq 0 \), one may easily check via the Dominated Convergence Theorem that \( \Lambda^b_{\varphi_\alpha, 1} \) is continuous on \( \mathbb{H} \).
In order to complete the proof of (a), we need to show continuity of $\Lambda_{\varphi,1}^b$ at $\infty$.

Since $\Lambda_{\varphi,1}^b(\infty) = 0$ by definition, we need to show

$$\Lambda_{\varphi,1}^b(x + iy) \to 0$$

as $|x| + y \to \infty$. To prove this, it suffices to show that (3.22) holds in each of the following three cases:

(i) $y \to \infty$ and $|x|$ stays bounded;
(ii) $x \to -\infty$;
(iii) $x \to \infty$.

First, consider Case (i). Let $M > 0$ and assume $|x| \leq \frac{\alpha}{2}$. For $t \geq M$, we have

$$t^\alpha - x \geq t^\alpha/2$$

and thus

$$\frac{1}{|t^\alpha - x + i(y + 1)|} \leq \frac{2}{|t^\alpha + 2i|}$$

for all $y \geq 0$. Meanwhile, for $0 \leq t \leq M$, we have

$$\frac{1}{|t^\alpha - x + i(y + 1)|} \leq 1$$

for all $y \geq 0$. We thus have by the Dominated Convergence Theorem

$$\lim_{y \to \infty} \Lambda_{\varphi,1}^b(x + iy) = 0,$$

as required.

Next, consider Case (ii). For $x \leq 0$, note

$$\frac{1}{|t^\alpha - x + i(y + 1)|} \leq \frac{1}{|t^\alpha + i|}$$

Thus, again by the Dominated Convergence Theorem, we obtain

$$\lim_{x \to -\infty} \Lambda_{\varphi,1}^b(x + iy) = 0,$$

as required.

Finally, consider Case (iii). Note

$$\Lambda_{\varphi,1}^b(x + iy) \leq \Lambda_{\varphi,1}^b(x) \approx \int_0^\infty \frac{dt}{|t^\alpha - x| + 1}$$

for all $x$ and $y$. We claim

$$I_\alpha(x) := \int_1^\infty \frac{dt}{|t^\alpha - x| + 1} \to 0$$

as $x \to \infty$. Note

$$\lim_{x \to -\infty} \int_0^1 \frac{dt}{|t^\alpha - x| + 1} = 0$$

by the Dominated Convergence Theorem. Thus, with (3.26) granted, we have $\Lambda_{\varphi,1}^b(x) \to 0$ as $x \to \infty$. This, together with (3.25), yields

$$\Lambda_{\varphi,1}^b(x + iy) \to 0$$

uniformly in $y$ as $x \to \infty$. So, combining (3.23), (3.24) and (3.27), we conclude (3.22), as required.

It remains to prove (3.26). First, we consider the case $\alpha \geq 2$. Note

$$I_\alpha(x) = \frac{2}{\alpha} \int_1^\infty \frac{ds}{(s^2 - x) + |s^{1-\frac{\alpha}{2}}|} \leq \int_0^{\infty} \frac{ds}{|s^2 - x| + 1} = I_2(x).$$
In conjunction with this, we note

\[
I_2(x^2) = \int_0^\infty \frac{ds}{|s^2 - x^2| + 1} = \int_0^{x-1} + \int_{x-1}^{x+1} + \int_{x+1}^{\infty}
\]

for \( x \geq 1 \). For the first integral of the above, we have

\[
\int_0^{x-1} \leq \int_0^{x-1} \frac{ds}{x^2 - s^2} = \frac{\log(2x-1)}{2x} \rightarrow 0
\]
as \( x \rightarrow \infty \). For the second integral, we have

\[
\int_{x-1}^{x+1} = \int_{-1}^{1} \frac{dr}{|2xr + 1| + 1} \rightarrow 0
\]
as \( x \rightarrow \infty \) by the Dominated Convergence Theorem. For the third integral, we have

\[
\int_{x+1}^{\infty} \leq \int_{x+1}^{\infty} \frac{ds}{s^2 - x^2} = \frac{\log(2x+1)}{2x} \rightarrow 0
\]
as \( x \rightarrow \infty \), which completes the proof of (3.26) for \( \alpha \geq 2 \).

Next, consider the case \( 1 < \alpha < 2 \). Note

\[
I_\alpha(x^2) = \int_{\sqrt{x}}^{\infty} \frac{dt}{|t^\alpha - x| + 1}.
\]

For the first integral, we have

\[
\int_{\sqrt{x}}^{1} \leq \int_{1}^{\sqrt{x}} \frac{dt}{x - t^2 + 1} \leq I_2(x) \rightarrow 0
\]
as \( x \rightarrow \infty \). For the second integral, we have

\[
\int_{\sqrt{x}}^{x^{\frac{1}{2}}} = \frac{1}{\alpha} \int_{x}^{x} \frac{ds}{(x - s + 1)s^{1 - \frac{1}{\alpha}}} \leq \frac{\log \left( x - x^{\frac{2}{\alpha}} + 1 \right)}{\alpha x^{\frac{\alpha - 1}{2}}} \rightarrow 0
\]
as \( x \rightarrow \infty \) by the Dominated Convergence Theorem. Thus, (3.26) also holds for \( 1 < \alpha < 2 \). This establishes (c), and completes the proof.

We note from Example 3.12 that \( S_{b_1}^h \) is bounded on \( L^1(\mathbb{R}) \), but \( S_{b_0}^h \) is not. So, the converse of Corollary 3.3 does not hold. In connection with this observation we notice that there exists in fact a much stronger example as in Example 3.14 below.

To construct such an example, we need a technical lemma; recall that the definition of the sets \( D_k(x) \) was given in (2.8).

Lemma 3.13. Let

\[
z_k = \frac{1}{2x} \left[ 1 + \frac{i}{(\log k)^2} \right], \quad k = 2, 3, \ldots
\]

Then there is an absolute constant \( C > 0 \) such that

\[
\sum_{k=M}^{\infty} \frac{\mathcal{K}^h(z_k, w)}{k2^k} \leq \frac{C}{\log M}, \quad w \in D_{2^{\alpha - 1}}(0)
\]

for positive integers \( M \geq 2 \).
Proof. Let \( M \geq 2 \) be an integer and \( w \in D_{2^{-M-1}}(0) \). Let \( j \geq M + 1 \) be the integer such that \( w \in D_{2^{-j}}(0) \setminus D_{2^{-j-1}}(0) \).

First, we show

\[
(3.28) \quad \sum_{k=M}^{j-1} \frac{\mathcal{K}^h(z_k, w)}{k^{2^k}} \leq \frac{C}{\log M}
\]

for some absolute constant \( C > 0 \). To see this, for \( M \leq k \leq j - 1 \), note

\[
|z_k - w| \geq |w| - |z_k| \geq \frac{1}{2^j} - \frac{1}{2^k} \geq \frac{1}{2^k+1}
\]

so that

\[
\mathcal{K}^h(z_k, w) = \frac{\text{Im} z_k + \text{Im} w}{\pi |z_k - w|^2} < 2^{2k+2} \left( \frac{1}{2^k(\log k)^2} + \frac{1}{2^j} \right).
\]

It follows that

\[
\sum_{k=M}^{j-1} \frac{\mathcal{K}^h(z_k, w)}{k^{2^k}} \leq \sum_{k=M}^{j-1} \frac{1}{k^{2^k} k (\log k)^2} + \frac{1}{2^j} \sum_{k=M}^{j-1} \frac{2^k}{k} \leq \frac{1}{\log M} + \frac{1}{M},
\]

which yields (3.28).

Next, we show that there is an absolute constant \( C > 0 \) such that

\[
(3.29) \quad \sum_{k=j+3}^{\infty} \frac{\mathcal{K}^h(z_k, w)}{k^{2^k}} \leq \frac{C}{j}.
\]

To see this, for \( k \geq j + 3 \), note

\[
|z_k - w| \geq |w| - |z_k| \geq \frac{1}{2^j+1} - \frac{1}{2^k} \geq \frac{1}{2^k+2}
\]

so that

\[
\mathcal{K}^h(z_k, w) \leq \frac{1}{|z_k - w|} \leq 2^{2+j}.
\]

Also, note

\[
\sum_{k=j+3}^{\infty} \frac{1}{k^{2^k}} \approx \frac{1}{j^{2^j}}.
\]

Thus, (3.29) holds.

Finally, since

\[
\mathcal{K}^h(z_k, w) \leq \frac{1}{\pi \text{Im} z_k} = \frac{2^k(\log k)^2}{\pi},
\]

there is an absolute constant \( C > 0 \) such that

\[
(3.30) \quad \sum_{k=j}^{j+2} \frac{\mathcal{K}^h(z_k, w)}{k^{2^k}} \leq \frac{C (\log j)^2}{j}
\]

for all integers \( j \geq 2 \).

Now, since \( j \geq M + 1 \), we conclude the lemma by (3.28), (3.29) and (3.30). The proof is complete. \( \square \)

**Example 3.14.** There is \( \varphi \in \mathcal{S}(\mathcal{H}) \) such that \( S^h_\varphi \) is compact on \( L^1(\mathbb{R}) \), but \( S^h_\varphi \) is not even bounded on \( L^1(\mathbb{R}) \).
Proof. Let \( \{z_k\}_{k=2}^\infty \) be the sequence specified in Lemma 3.13. Let \( \{E_k\}_{k=2}^\infty \) be a partition of the interval \([-1, 1]\) into Borel sets \( E_k \) such that \( m(E_k) = \frac{c}{k^{2k}} \), where \( c^{-1} = \frac{1}{2} \sum_{k=2}^\infty \frac{1}{k^{2k}} \). Define \( \varphi: \mathbb{R} \to \mathbb{H} \) by

\[
\varphi(x) := \begin{cases} 
  z_k & \text{if } x \in E_k \text{ for } k = 2, 3, \ldots \\
  x^2i & \text{if } x \in A 
\end{cases}
\]

where \( A := \mathbb{R} \setminus [-1, 1] \). Clearly, \( \varphi \in S(\mathbb{H}) \).

Given a Borel set \( E \subset \mathbb{H} \), note

\[
\varphi^{-1}(E) = \left( \bigcup_{z_k \in E} E_k \right) \bigcup \psi^{-1}(E)
\]

where \( \psi: A \to \mathbb{H} \) is the function \( x \mapsto x^2i \). It follows that

\[
\mu_\varphi = c \sum_{k=2}^\infty \frac{\delta_k}{k^{2k}} + m \circ \psi^{-1}
\]

where \( \delta_k \) is the unit point mass at \( z_k \). Thus

\[
\Lambda_{\varphi, 1}(w) = c \sum_{k=2}^\infty \frac{1}{k^{2k}} |\mathcal{K}^\sigma(w, z_k)| + \int_{|x|>1} |\mathcal{K}^\sigma(w, x^2i)| \, dx
\]

\[
= cF^\sigma(w) + G^\sigma(w)
\]

for \( w \in \mathbb{H} \) and \( \sigma \in \{b, h\} \). Note

\[
F^b(0) = \frac{1}{2\pi} \sum_{k=2}^\infty \frac{1}{k^{2k}} \frac{1}{|z_k|} = \frac{1}{2\pi} \sum_{k=2}^\infty \frac{k|1 + (\log k)^{-2}i|}{k^{2k}} = \infty.
\]

Thus, from Fatou’s Lemma and Theorem 3.1, \( S^b_\varphi \) is not bounded on \( L^1(\mathbb{R}) \).

We now turn to the proof that \( S^h_\varphi \) is compact on \( L^1(\mathbb{R}) \). Since \( x^{-2} \) is integrable near \( \infty \), it is easily verified by the Dominated Convergence Theorem that \( G^h \in C(\mathbb{H}) \) with \( G^h(\infty) = 0 \). Thus, by Theorem 3.9, it suffice to show \( F^h \in C(\mathbb{H}) \) with \( F^h(\infty) = 0 \).

First, since \( z_k \to 0 \), it is not hard to see that the series in (3.31) converges uniformly on each compact subset of \( \mathbb{H} \setminus \{0\} \). So, \( F^h \) is continuous on \( \mathbb{H} \setminus \{0\} \).

Next, note

\[
|\mathcal{K}^h(w, z_k)| \leq \frac{1}{\pi \text{Im } w}
\]

for all \( k \) and \( w \in \mathbb{H} \). Also, when \( \text{Im } w \) stays bounded, note

\[
|\mathcal{K}^h(w, z_k)| \leq \frac{1}{|w - z_k|^2} \leq \frac{1}{|w|^2}
\]

for all \( k \) and for all \( w \in \mathbb{H} \) with \( |w| \) sufficiently large. It follows that \( F^h(w) \to 0 \) as \( w \to \infty \). This implies that \( F^h \) is continuous at \( \infty \) with \( F^h(\infty) = 0 \).

Finally, it remains to show that \( F^h \) is continuous at 0. Let \( M \) be a large positive integer. We note

\[
|F^h(w) - F^h(0)| \leq \sum_{k=2}^{M-1} |\mathcal{K}^h(w, z_k)| + \sum_{k=M}^\infty \frac{|\mathcal{K}^h(w, z_k) - \mathcal{K}^h(0, z_k)|}{k^{2k}}
\]
For the first sum, it is clear that \( \lim_{w \to 0} \sum_{k=2}^{M-1} = 0 \). For the second sum, note from Lemma 3.13
\[
\limsup_{w \to 0} \sum_{k=M}^{\infty} \frac{X^h(w, z_k)}{k2^k} \leq \frac{C}{\log M}
\]
for some absolute constant \( C \). Meanwhile, since
\[
X^h(0, z_k) = \frac{\Im z_k}{\pi |z_k|^2} \approx \frac{2^k}{(\log k)^2},
\]
we have
\[
\sum_{k=M}^{\infty} \frac{X^h(0, z_k)}{k2^k} \approx \sum_{k=M}^{\infty} \frac{1}{k(\log k)^2} \approx \frac{1}{\log M};
\]
constants suppressed above are independent of \( M \). Combining these observations, we deduce
\[
\limsup_{w \to 0} |F^h(w) - F^h(0) | \leq \frac{C}{\log M}
\]
for some constant \( C > 0 \) independent of \( M \). Thus, taking the limit \( M \to \infty \), we conclude continuity of \( F^h \) at 0. The proof is complete. □

4. \( L^p \)-Characterizations

In this section we prove more detailed versions of Theorem 1.2 and related facts. We assume \( 1 < p < \infty \), unless otherwise specified, throughout the section.

The key tool to our proofs is the notion of \( L^p_\sigma \)-Carleson measures, \( \sigma \in \{ b, h \} \), which were described in Section 2.4. Recall that the notion of \( L^p_\sigma \)-Carleson measures does not depend on \( p \) and \( \sigma \). So, in what follows, a (respectively compact) \( L^p_\sigma \)-Carleson measure will be simply called a (compact) Carleson measure.

Our starting point is the next lemma, which is a consequence of (2.9) and (2.11).

**Lemma 4.1.** Let \( \varphi \in \mathcal{S}(H) \), \( \sigma \in \{ b, h \} \) and \( 1 < p < \infty \). Then \( S^\sigma_\varphi \) is bounded (respectively compact) on \( L^p(R) \) if and only if \( \mu_\varphi \) is a (compact) Carleson measure. Moreover, the operator norm satisfies
\[
\| S^\sigma_\varphi \|_{L^p(R)} \approx M(\mu_\varphi);
\]
the constants suppressed above depend on \( \sigma \) and \( p \), but are independent of \( \varphi \).

Before proceeding, we notice as an immediate consequence Lemma 4.1 that \( L^1 \)-boundedness/compactness of the operators implies \( L^p \)-boundedness/compactness. For the failure of the converse, see Example 4.8 at the end of the section.

**Corollary 4.2.** Let \( \varphi \in \mathcal{S}(H) \), \( \sigma \in \{ b, h \} \) and \( 1 < p < \infty \). If \( S^\sigma_\varphi \) is bounded (respectively compact) on \( L^1(R) \), then \( S^\sigma_\varphi \) is bounded (compact) on \( L^p(R) \).

**Proof.** We first consider the case \( \sigma = b \). Suppose that \( S^b_\varphi \) is bounded (respectively compact) on \( L^1(R) \). Then \( S^b_\varphi : H^1(R) \to L^1(R) \) is bounded (compact). So, we see from (2.11) and the remark at the end of Section 2.4 that \( \mu_\varphi \) is a (compact) Carleson measure. Thus \( S^b_\varphi \) is bounded (compact) on \( L^p(R) \) by Lemma 4.1.

Note by the reproducing properties of the Borel kernel and the Poisson kernel that \( S^b_\varphi = S^h_\varphi \) on \( H^1(R) \). So, the same proof works for the case \( \sigma = h \). The proof is complete. □

We also need a couple of auxiliary estimates.
Lemma 4.3. Given \( \sigma \in \{b, h\} \) and \( 1 < p < \infty \), the equality
\[
\| K^\sigma(z, \cdot) \|_p = (\text{Im } z)^{\frac{1}{p} - 1} \| K^\sigma(i, \cdot) \|_p
\]
holds for \( z \in \mathbb{H} \).

Proof. The lemma holds by the elementary integral identity
\[
\int_{-\infty}^{\infty} dt = \frac{1}{(\text{Im } z)^c} \int_{-\infty}^{\infty} ds (1 + s^2)^{1+c}
\]
valid for \( z \in \mathbb{H} \) and \( c \) real. \( \square \)

Recall that \( D_\delta(x) \) below denotes the Carleson set introduced in (2.8).

Lemma 4.4. Let \( \varphi \in S(\mathbb{H}) \), \( \sigma \in \{b, h\} \) and \( 1 \leq p < \infty \). Then there is a constant \( C = C(\sigma, p) > 0 \) such that
\[
[\Lambda^\sigma_{\varphi, p}(x + \delta i)]^p \geq C \frac{\mu_\varphi(D_\delta(x))}{\delta}
\]
for \( x \in \mathbb{R} \) and \( \delta > 0 \).

Proof. By (2.2) we only need to consider \( \sigma = h \). Let \( x \in \mathbb{R} \), \( \delta > 0 \) and \( w \in D_\delta(x) \). We have
\[
|x + \delta i - \overline{w}| \leq |x - w| + \delta < 2\delta
\]
and hence
\[
|\mathcal{K}^h_{\varphi}(w, x + \delta i)| = \frac{1}{\pi} \frac{\text{Im } w + \delta}{|x + \delta i - \overline{w}|^2} > \frac{1}{4\pi\delta}.
\]
Thus, when \( 1 < p < \infty \), we have by (2.10) and Lemma 4.3
\[
[\Lambda^h_{\varphi, p}(x + \delta i)]^p = \frac{\delta^{p-1}}{\| K^\sigma(i, \cdot) \|_p} \int_{\mathbb{H}} |\mathcal{K}^h_{\varphi}(x + \delta i, w)|^p d\mu_\varphi(w)
\geq C \delta^{p-1} \int_{D_\delta(x)} \delta^{-p} d\mu_\varphi(w)
= C \frac{\mu_\varphi(D_\delta(x))}{\delta}
\]
for some constant \( C = C(p) > 0 \). The same estimate also holds for \( p = 1 \) by an easy modification. The proof is complete. \( \square \)

We are now ready to characterize boundedness for the case \( 1 < p < \infty \). The next theorem is a more detailed version of the boundedness part of Theorem 1.2.

Theorem 4.5. Let \( \varphi \in S(\mathbb{H}) \) and \( \sigma \in \{b, h\} \). Then the following statements are equivalent:

(a) \( S^\sigma_\varphi \) is bounded on \( L^p(\mathbb{R}) \) for some/all \( p \in (1, \infty) \);
(b) \( \Lambda^\sigma_{\varphi, p} \) is bounded on \( \mathbb{H} \) for some/all \( p \in (1, \infty) \);
(c) \( \mu_\varphi \) is a Carleson measure.

Moreover, the operator norm satisfies
\[
\| S^\sigma_\varphi \|_{L^p(\mathbb{R})} \approx \| \Lambda^\sigma_{\varphi, p} \|_{\mathbb{H}} \approx \| M(\mu_\varphi) \|^{\frac{1}{p}};
\]
the constants suppressed above depend on \( \sigma \) and \( p \), but are independent of \( \varphi \).

Note that \( p \) and \( \sigma \) do not appear in statement (c). Hence, as mentioned in the Introduction, the boundedness of \( S^\sigma_\varphi \) on \( L^p(\mathbb{R}) \) is independent of \( \sigma \in \{b, h\} \) and \( p \in (1, \infty) \).
Proof. Let \( 1 < p < \infty \). By Lemma 4.1 it suffices to show
\[
C^{-1} M(\mu_{\varphi}) \leq \|\Lambda_{\varphi,p}^\sigma\|_H \leq CM(\mu_{\varphi}).
\]
for some constant \( C = C(\sigma, p) > 0 \).

The first inequality holds by Lemma 4.4. Assume \( M(\mu_{\varphi}) < \infty \), or equivalently by Lemma 4.1, that \( S_{\varphi}^\sigma \) is bounded on \( L^p(\mathbb{R}) \). As in the proof of Theorem 3.1, choosing again the Poisson kernels as test functions, we have by (3.2)
\[
\|S_{\varphi}^\sigma\|_{L^p(\mathbb{R})} \geq \frac{\|\mathcal{K}^\sigma(z, \varphi(\cdot))\|_p}{\|K^\sigma(z, \cdot)\|_p} = \Lambda_{\varphi,p}^\sigma(z)
\]
for any \( z \in \mathbb{H} \) and thus obtain
\[
\|S_{\varphi}^\sigma\|_{L^p(\mathbb{R})} \geq \|\Lambda_{\varphi,p}^\sigma\|_H.
\]
This, together with Lemma 4.1, yields the second inequality in (4.1). The proof is complete. \( \Box \)

Remark. Note that the first inequality in (4.1) remains valid for \( p = 1 \) by Lemma 4.4. Thus, when \( S_{\varphi}^\sigma \) bounded on \( L^1(\mathbb{R}) \), we obtain by Theorem 4.5 the operator norm estimate
\[
\|S_{\varphi}^\sigma\|_{L^p(\mathbb{R})} \lesssim \|S_{\varphi}^\sigma\|_{L^1(\mathbb{R})}.
\]
In fact, in case \( \sigma = h \), a much better inequality holds as follows. Note that \( S_{\varphi}^h \) clearly acts boundedly on \( L^\infty(\mathbb{R}) \) with norm 1. Thus one may use the Riesz-Thorin Interpolation Theorem to obtain
\[
\|S_{\varphi}^h\|_{L^p(\mathbb{R})} \lesssim \|S_{\varphi}^h\|_{L^1(\mathbb{R})}. \tag{4.2}
\]
Here is an elementary proof: (Proof) Note that \( \mathcal{K}^h(t + iy, \varphi(x)) dt \) is a probability measure for each \( x \in \mathbb{R} \) and \( y > 0 \). So, given \( f \in L^p(\mathbb{R}) \), applications of (1.4), Fatou’s Lemma and Jensen’s Inequality yield
\[
\|S_{\varphi}^h f\|_p^p \leq \liminf_{y \downarrow 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t)|^p \mathcal{K}^h(t + iy, \varphi(x)) dt dx.
\]
Now, computing the x-integration first and applying Theorem 3.1, we obtain
\[
\|S_{\varphi}^h f\|_p^p \leq \|f\|_p^p \Lambda_{\varphi,1}^h \|H = \|f\|_p^p \|S_{\varphi}^h\|_{L^1(\mathbb{R})},
\]
which yields (4.2).

We now prove the following more detailed version of the compactness part of Theorem 1.2.

**Theorem 4.6.** Let \( \varphi \in S(\mathbb{H}) \) and \( \sigma \in \{b, h\} \). Then the following statements are equivalent:

(a) \( S_{\varphi}^\sigma \) is compact on \( L^p(\mathbb{R}) \) for some/all \( p \in (1, \infty) \);
(b) \( \Lambda_{\varphi,p}^\sigma \in C_0(\hat{\mathbb{H}}) \) for some/all \( p \in (1, \infty) \);
(c) \( \mu_{\varphi} \) is a compact Carleson measure.

Again, notice that it follows that compactness of \( S_{\varphi}^\sigma \) on \( L^p(\mathbb{R}) \) is independent of \( \sigma \in \{b, h\} \) and \( p \in (1, \infty) \).
Proof. It suffices to show
\[ \Lambda^\sigma_{\varphi,p} \in C_0(\hat{H}) \iff \mu_{\varphi} \text{ is a compact Carleson measure} \]
by Lemma 4.1.

Assume \( \Lambda^\sigma_{\varphi,p} \in C_0(\hat{H}) \). Note from Lemma 4.1 and Theorem 4.5 that \( \mu_{\varphi} \) is a Carleson measure. We have by Lemma 4.4 that
\[ M_\delta(\mu_{\varphi}) \lesssim \sup_{x \in \mathbb{R}} \Lambda^\sigma_{\varphi,p}(x + \delta i) \]
for \( \delta > 0 \). Note that the right-hand side of the above tends to 0 as \( \delta \downarrow 0 \), from the assumption that \( \Lambda^\sigma_{\varphi,p} \in C_0(\hat{H}) \). Thus \( \mu_{\varphi} \) is a compact Carleson measure on \( \hat{H} \), as required.

Conversely, assume that \( \mu_{\varphi} \) is a compact Carleson measure on \( \hat{H} \), or equivalently by Lemma 4.1, that \( S^\sigma_{\varphi} \) is compact on \( L^p(\mathbb{R}) \). Note
\[ \Lambda^\sigma_{\varphi,p}(z) = \|S^\sigma_{\varphi} k^\sigma_z \|_p, \quad z \in H \]
where \( k^\sigma_z := \mathcal{K}^\sigma(z, \cdot)/\|K^\sigma(z, \cdot)\|_p \). Using this and \( L^p \)-boundedness of \( S^\sigma_{\varphi} \), one may check that \( \Lambda^\sigma_{\varphi,p} \in C(\mathbb{H}) \). Meanwhile, note from Lemma 4.3
\[ S^\sigma_{\varphi} k^\sigma_z(t) = (\text{Im } z)^{1-p} \frac{\mathcal{K}^\sigma(z, \varphi(t))}{\|K^\sigma(t, \cdot)\|_p}. \]
Hence \( S^\sigma_{\varphi} k^\sigma_z \to 0 \) pointwise on \( \mathbb{R} \setminus \varphi^{-1}(\mathbb{R}) \) as \( \text{Im } z \to 0 \) or \( z \to \infty \). Next we show that \( m[\varphi^{-1}(\mathbb{R})] = \mu_{\varphi}(\mathbb{R}) = 0 \). To see this, observe that our assumption that \( \mu_{\varphi} \) is a compact Carleson measure implies that the symmetric derivative of \( \mu_{\varphi} \) vanishes everywhere on \( \mathbb{R} \), i.e.,
\[ \lim_{\delta \to 0} \mu_{\varphi}[D_\delta(x) \cap \mathbb{R}] = 0 \]
for all \( x \in \mathbb{R} \). From this, using \([14, \text{Theorems 7.14 and 7.15}] \) it follows that the restriction of \( \mu_{\varphi} \) to \( \mathbb{R} \) is the zero measure. Thus \( m[\varphi^{-1}(\mathbb{R})] = 0 \), and hence \( S^\sigma_{\varphi} k^\sigma_z \to 0 \) pointwise almost everywhere on \( \mathbb{R} \) as \( \text{Im } z \to 0 \) or \( z \to \infty \). Now, noting \( \|k^\sigma_z\|_p = 1 \) and using the compactness of \( S^\sigma_{\varphi} \), one may deduce that \( \Lambda^\sigma_{\varphi,p} \in C_0(\hat{H}) \). The proof is complete. \( \square \)

As a consequence, we obtain the next corollary, which is an analogue of Lemma 3.7.

**Corollary 4.7.** Let \( \varphi \in S(\mathbb{H}) \), \( \sigma \in \{b, h\} \) and \( 1 < p < \infty \). Assume \( \Lambda^\sigma_{\varphi,1} \in C(\hat{H}) \) and \( \varphi(\mathbb{R}) \subset \mathbb{H} + \epsilon i \) for some \( \epsilon > 0 \). Then \( S^\sigma_{\varphi} \) is compact on \( L^p(\mathbb{R}) \).

**Proof.** Since \( \Lambda^\sigma_{\varphi,1} \in C(\hat{H}) \) by assumption, we note by Theorem 3.1 and Corollary 4.2 that \( S^\sigma_{\varphi} \) is bounded on \( L^p(\mathbb{R}) \). Thus \( \Lambda^\sigma_{\varphi,p} \in C(\mathbb{H}) \) as in the proof of Theorem 4.6.

Since \( \varphi(\mathbb{R}) \subset \mathbb{H} + \epsilon i \) by assumption, we note that \( \mu_{\varphi} \) is supported on \( \mathbb{H} + \epsilon i \). Thus we have by Lemma 4.3 and (2.10)
\[ [\Lambda^\sigma_{\varphi,p}(z)]^p \|K^\sigma(t, \cdot)\|_p = (\text{Im } z)^{p-1} \int_{\mathbb{H} + \epsilon i} |\mathcal{K}^\sigma(z, w)|^p \, d\mu_{\varphi}(w) \]
for \( z \in H \). Thus, using the inequality \( |\mathcal{K}(z, w)| \leq \left( \frac{2\pi \text{Im} w}{\pi} \right)^{\frac{1}{p}} \), we see that
\[
\Lambda_{\sigma}^{\varphi, p}(z)^p \| K(i, \cdot) \|_{p}^{p} \leq \left( \frac{\text{Im} z}{2\pi \epsilon} \right)^{\frac{1}{p}} \Lambda_{\varphi, 1}^{\sigma}(z),
\]
which yields
\[
\lim_{\text{Im} z \downarrow 0} \Lambda_{\phi, p}^{\sigma}(z) = 0. \tag{4.3}
\]
Similarly, using the inequality \( |\mathcal{K}(z, w)| \leq \left( \frac{2\pi \text{Im} z}{\pi} \right)^{\frac{1}{p}} \), we obtain
\[
\Lambda_{\varphi, 1}^{\sigma}(z) \leq \left( \frac{1}{2\pi} \right)^{\frac{1}{p}} \Lambda_{\varphi, 1}^{\sigma}(z).
\]
Thus, we deduce from continuity of \( \Lambda_{\varphi, 1}^{\sigma} \) at \( \infty \) that
\[
\lim_{z \to \infty} \Lambda_{\phi, p}^{\sigma}(z) = 0. \tag{4.4}
\]
Finally, we conclude \( \Lambda_{\varphi, p}^{\sigma} \in C_{0}(\hat{H}) \) by (4.3) and (4.4). So, \( S_{\varphi}^{\sigma} \) is compact on \( L^{p}(R) \) by Theorem 4.6. The proof is complete. \( \square \)

Finally, applying our characterizations, we show by explicit examples that \( L^{p} \)-compactness may not imply \( L^{1} \)-boundedness. This, in particular, shows that the converse of Corollary 4.2 fails.

**Example 4.8.** Let \( \varphi \in \mathcal{S}(H) \) be the function defined by
\[
\varphi(x) := \begin{cases} 
0 & \text{if } x = 0 \\
|x|(1 - \log |x|)i & \text{if } 0 < |x| \leq 1 \\
x^{2}i & \text{if } |x| > 1
\end{cases}.
\]
Then, for each \( \sigma \in \{b, h\} \), \( S_{\varphi}^{\sigma} \) is compact on \( L^{p}(R) \), \( 1 < p < \infty \), but \( S_{\varphi}^{\sigma} \) is not bounded on \( L^{1}(R) \).

**Proof.** By Corollary 3.3 and Lemma 4.1 we may assume \( \sigma = h \). Since \( \text{Im} \varphi = |\varphi| \), we see that the function
\[
\mathcal{K}^{h}(0, \varphi(\cdot)) = \frac{1}{|\varphi(\cdot)|}
\]
is not integrable near 0, i.e., \( \Lambda_{\varphi, 1}^{h}(0) = \infty \). Thus \( S_{\varphi}^{h} \) is not bounded on \( L^{1}(R) \) by Fatou’s Lemma and Theorem 3.1.

To prove \( L^{p} \)-compactness of \( S_{\varphi}^{h} \), it suffices by Lemma 4.1 to show that \( \mu_{\varphi} \) is a compact Carleson measure. Given \( \zeta \in R \), note
\[
|\varphi(x)| \leq |\zeta - \varphi(x)|, \quad x \in R
\]
so that
\[
\varphi^{-1}[D_{\delta}(\zeta)] \subset \varphi^{-1}[D_{\delta}(0)]
\]
for any \( \delta > 0 \). Also, note that the function \( \eta := |\varphi| \) is strictly increasing on \([0, \infty)\). We thus have
\[
\varphi^{-1}[D_{\delta}(0)] = (-\eta^{-1}(\delta), \eta^{-1}(\delta))
\]
and hence
\[
M_{\delta}^{\mu_{\varphi}} = \frac{\mu_{\varphi}[D_{\delta}(0)]}{\delta} = \frac{2\eta^{-1}(\delta)}{\delta},
\]
for all \( \delta > 0 \). It is now elementary to see that \( M_\delta(\mu_\varphi) \) is bounded, i.e. \( \mu_\varphi \) is a Carleson measure, and also that \( M_\delta(\mu_\varphi) \to 0 \) as \( \delta \downarrow 0 \). Thus, \( \mu_\varphi \) is a compact Carleson measure and the proof is complete. \hfill \Box

5. Holomorphic Symbols

In this section we apply our \( L^1 \)-characterization to the case of holomorphic symbols and obtain two additional characterizations for the operators associated with the Poisson kernel. So, throughout the section, we restrict ourselves to symbols \( \varphi \) given by the boundary function of a holomorphic self map of \( \mathbb{H} \); i.e. \( \varphi \in \mathcal{S}_a(\mathbb{H}) \).

Recall from Section 2.6 that \( \mathcal{S}_a(\mathbb{H}) \subset \mathcal{S}(\mathbb{H}) \). We will freely identify a function \( \varphi \in \mathcal{S}_a(\mathbb{H}) \) with its holomorphic extension to \( \mathbb{H} \). We will also freely identify the spaces \( H^p(\mathbb{R}) \) with \( H^p(\mathbb{H}) \), \( 1 \leq p < \infty \); see the comments at the end of Section 2.3.

For \( \varphi \in \mathcal{S}_a(\mathbb{H}) \), define

\[
D(\varphi) := \sup_{z \in \mathbb{H}} \frac{\text{Im} \, z}{\text{Im} \, \varphi(z)}.
\]

In [8] Elliott and Jury proved a half-plane version of the Julia-Carathéodory Theorem showing that \( D(\varphi) \) is the angular derivative of \( \varphi \) at infinity (when \( \varphi \) fixes infinity), and moreover that the norm of \( C_\varphi \) on \( H^2(\mathbb{R}) \) is \( \sqrt{D(\varphi)} \). In the theorem below, we show that \( D(\varphi) < \infty \) also characterizes when \( S^b_\varphi \) is bounded on \( L^1(\mathbb{R}) \).

While the “only if” part of the theorem is an easy consequence of the Elliott-Jury result, we include a short proof using our methods.

For fixed \( w \in \mathbb{H} \), note that \( \mathcal{K}^h(w, \varphi(z)) \) is a bounded harmonic function of the variable \( z \in \mathbb{H} \). Hence, by (2.6), it is the Poisson integral of its boundary function, i.e.,

\[
(5.1) \quad \mathcal{K}^h(w, \varphi(z)) = \int_{-\infty}^{\infty} P_z(t) \mathcal{K}^h(w, \varphi(t)) \, dt, \quad z \in \mathbb{H};
\]

recall that \( P_z \) denotes the Poisson kernel specified in (2.3).

Our first result is the following characterization in terms of \( D(\varphi) \).

**Theorem 5.1.** Let \( \varphi \in \mathcal{S}_a(\mathbb{H}) \). Then \( S^b_\varphi \) is bounded on \( L^1(\mathbb{R}) \) if and only if \( D(\varphi) < \infty \). Moreover, \( \| S^b_\varphi \|_{L^1} = D(\varphi) \).

**Proof.** Recalling from Theorem 3.1 that \( \| S^b_\varphi \|_{L^1} = \| \Lambda^b_{\varphi, 1} \|_{\mathbb{H}} \), it suffices to show that \( D(\varphi) = \| \Lambda^b_{\varphi, 1} \|_{\mathbb{H}} \). For \( y > 0 \), we have

\[
\int_{-\sqrt{y}}^{\sqrt{y}} \mathcal{K}^h(w, \varphi(t)) \, dt \leq (1 + y) \int_{-\sqrt{y}}^{\sqrt{y}} \frac{y}{t^2 + y^2} \mathcal{K}^h(w, \varphi(t)) \, dt
\]

\[
\leq \pi(1 + y) \int_{-\infty}^{\infty} P_{iy}(t) \mathcal{K}^h(w, \varphi(t)) \, dt
\]

\[
= \pi(1 + y) \mathcal{K}^h(w, \varphi(iy)) \quad \text{by (5.1)}
\]

\[
\leq \frac{1 + y}{\text{Im} \, \varphi(iy)}.
\]

Hence,

\[
\int_{-\infty}^{\infty} \mathcal{K}^h(w, \varphi(t)) \, dt \leq \lim_{y \to \infty} \frac{1 + y}{y} D(\varphi) = D(\varphi).
\]

Taking the supremum over \( w \in \mathbb{H} \) gives \( \| \Lambda^b_{\varphi, 1} \|_{\mathbb{H}} \leq D(\varphi) \).
Proposition 5.4. Suppose that $\varphi \in \mathcal{S}_a(\mathbb{H})$. When $\varphi / \varphi$ is compact on $H^1(\mathbb{R})$ or on $H^1(\mathbb{R})$, there is no $\varphi \in \mathcal{S}_a(\mathbb{H})$ such that $S^h_\varphi$ is compact on $L^1(\mathbb{R})$ or on $H^1(\mathbb{R})$. We show how this can be deduced from our work. To this end we note the following.

Lemma 5.2. Let $\varphi \in \mathcal{S}_a(\mathbb{H})$ and let $w \in \mathbb{H}$. Then, for all $y \geq 0$,

\[ \Lambda^h_{\varphi,1}(w) = \int_{-\infty}^{\infty} \mathcal{X}^h(w, \varphi(x + iy)) \, dx. \]

Proof. This is known that there are no $\varphi \in \mathcal{S}_a(\mathbb{H})$ such that $C^h_\varphi$ is compact on $H^1(\mathbb{R})$. This was first proved in [12]; see also [16] and [8] for extensions of this result. Hence, recalling from Section 2.6 that $S^h_\varphi$ coincides with the ordinary composition operator $C^h_\varphi$ on $H^1(\mathbb{R})$, there is no $\varphi \in \mathcal{S}_a(\mathbb{H})$ such that $S^h_\varphi$ is compact on $L^1(\mathbb{R})$ or on $H^1(\mathbb{R})$. We show how this can be deduced from our work. To this end we note the following.

Corollary 5.3. There is no $\varphi \in \mathcal{S}_a(\mathbb{H})$ such that $S^h_\varphi$ is compact on $L^1(\mathbb{R})$ or on $H^1(\mathbb{R})$. 

Proof. Suppose that $S^h_\varphi$ is bounded on $L^1(\mathbb{R})$, where $\varphi = u + iv \in \mathcal{S}_a(\mathbb{H})$. Then $D(\varphi) < \infty$ by Theorem 5.1. Showing that $S^h_\varphi$ is not compact on $L^1(\mathbb{R})$ will complete the proof. Let $b \geq 1$. By Lemma 5.2

\[ \Lambda^h_{\varphi,1}(ib) = \int_{-\infty}^{\infty} \mathcal{X}^h(ib, \varphi(x + iy)) \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{b + v(x + iy)}{u^2(x + iy) + (b + v(x + iy))^2} \, dx, \]

for all $y \geq 0$. Now take $y \geq D(\varphi) b$, so $v(x + iy) \geq b \geq 1$ for all $x \in \mathbb{R}$ from the definition of $D(\varphi)$. Thus

\[ \Lambda^h_{\varphi,1}(ib) \geq \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1 + v(x + iy)}{u^2(x + iy) + (1 + v(x + iy))^2} \, dx = \frac{1}{4} \Lambda^h_{\varphi,1}(i), \]

where Lemma 5.2 was used for the last equality. Since $\Lambda^h_{\varphi,1}(i) > 0$, it follows that $\lim_{y \to \infty} \Lambda^h_{\varphi,1}(ib) \neq 0$. Thus we see from Theorem 3.9 that $S^h_\varphi$ is not compact on $L^1(\mathbb{R})$, and this completes the proof. 

It is interesting to notice what Corollary 3.11 says in our current setting. When $\varphi \in \mathcal{S}_a(\mathbb{H})$ the operators $S^h_\varphi$ and $S^b_\varphi$ are equal on $H^1(\mathbb{R})$, and since $S^h_\varphi$ can not be compact it follows from Corollary 3.11 that $\frac{1}{\lambda_{\varphi}} \notin L^1(\mathbb{R})$. In fact more is true:

Proposition 5.4. If $\varphi \in \mathcal{S}_a(\mathbb{H})$, then $\frac{1}{\lambda_{\varphi}} \notin L^1(\mathbb{R})$. 

Proof. Let \( \varphi \in S_a(H) \) and put \( f := -\frac{1}{\varphi} \), so \( f \in S_a(H) \). If also \( f \in H^1(R) \), then
\[
\lim_{y \to \infty} \pi y \cdot \text{Im} f(iy) = \lim_{y \to \infty} y \int_{-\infty}^{\infty} \frac{y}{t^2 + y^2} \text{Im} f(t) dt = \|\text{Im} f\|_1 > 0,
\]
by the Dominated Convergence Theorem. Also, by the corollary after Theorem 10.1 in [7],
\[
g(\lambda) := f \circ \kappa(\lambda) \in H^1(D)
\]
where \( \kappa(\lambda) := \frac{1}{1 - \lambda} \). Hence
\[
|f \circ k(r)| = (1 - r) \frac{g(r)}{1 - r} \to 0,
\]
as \( r \uparrow 1 \). But, with \( y = (1 + r)/(1 - r) \), this contradicts (5.2). \( \square \)

Our second result is by means of the Nevanlinna representations, which is described below, of the holomorphic self-maps of \( H \). Let \( \psi \in S_a(H) \). Then, being positive and harmonic on \( H \), the function \( \text{Im} \psi \) admits its Herglotz representation
\[
\text{Im} \psi(z) = \beta \text{Im} z + \int_{-\infty}^{\infty} P_z(t) d\mu(t), \quad z \in H
\]
where \( \beta \geq 0 \) and \( \mu \geq 0 \) is a Borel measure on \( R \) such that \( (1 + t^2)^{-1} \in L^1(\mu) \). Noting
\[
P_z(t) = \frac{1}{\pi(1 + t^2)} \text{Im} \left( \frac{1 + tz}{t - z} \right),
\]
we may rephrase (5.3) as
\[
\text{Im} \psi(z) = \text{Im} \left\{ \beta z + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} \frac{d\mu(t)}{\pi(1 + t^2)} \right\}.
\]
So, we see that \( \psi \) is of the form
\[
\psi(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} d\rho(t)
\]
where \( \alpha \in R, \beta \geq 0 \) and \( \rho \geq 0 \) is a finite Borel measure on \( R \); this is called the Nevanlinna representation of \( \psi \).

For \( \psi \) as in (5.4), one may easily check in connection with Theorem 5.1 that \( D(\psi) < \infty \) if and only if \( \beta > 0 \), as is pointed out in [2]. Nevertheless, when \( \beta = 0 \), we still have the following characterization.

**Theorem 5.5.** Let \( \varphi \in S_a(H) \), and let \( \psi := -\frac{1}{\varphi} \) have Nevanlinna representation given in (5.4). Then \( S_h^b \) is bounded on \( L^1(R) \) if and only if the following statements all hold:
(a) \( \beta = 0 \);
(b) \( t^2 \in L^1(\rho) \) (and hence also \( t \in L^1(\rho) \));
(c) \( \int_{-\infty}^{\infty} t \, d\rho(t) = \alpha \).
Moreover, \( \|S_h^b\|_{L^1(R)} = \int_{-\infty}^{\infty} (1 + t^2) \, d\rho(t) \).

**Proof.** First assume that \( S_h^b \) is bounded on \( L^1(R) \). Then, by Theorem 5.1, \( D(\varphi) < \infty \) and
\[
\lim_{y \to \infty} \text{Im} \varphi(iy) \geq \lim_{y \to \infty} \frac{y}{D(\varphi)} = \infty.
\]
Thus $\psi(iy) = -1/\phi(iy) \to 0$ as $y \to \infty$, and from this it is clear that $\beta = 0$ so statement (a) holds.

Next, from the Nevanlinna representation for $\psi$ with $\beta = 0$,

$$
\psi(iy) = \alpha + \int_{-\infty}^{\infty} \frac{1 + ity}{t - iy} \, d\rho(t) = \alpha + \int_{-\infty}^{\infty} \left( \frac{1 + t^2}{t - iy} - t \right) \, d\rho(t).
$$

Note $(\Im \phi)(\Im \psi) \leq 1$. Thus, for any $\epsilon > 0$ and $y > 0$,

$$
D(\varphi) \geq y \Im \psi(iy) = \int_{-\infty}^{\infty} \frac{y^2}{t^2 + y^2} (1 + t^2) \, d\rho(t) \geq \frac{1}{1 + \epsilon^2} \int_{-\epsilon y}^{\epsilon y} (1 + t^2) \, d\rho(t).
$$

Now, taking the limit $y \to \infty$ and then $\epsilon \to 0$, we obtain

$$
D(\varphi) \geq \int_{-\infty}^{\infty} (1 + t^2) \, d\rho(t),
$$

showing that statement (b) holds.

To prove statement (c), note from (5.5) that

$$
\lim_{y \to \infty} \psi(iy) = \lim_{y \to \infty} -\frac{1}{\varphi(iy)} = 0.
$$

Hence, letting $y \to \infty$ in (5.6) and using the Dominated Convergence Theorem, we get that

$$
0 = \alpha - \int_{-\infty}^{\infty} t \, d\rho(t),
$$

which is statement (c).

Turning to the converse, assume statements (a), (b) and (c) all hold. For $z = x + iy \in \mathbb{H}$, from the Nevanlinna representation and using (a) and (c), we have

$$
\psi(z) = \int_{-\infty}^{\infty} \left( \frac{1 + tz}{t - z} + t \right) \, d\rho(t) = \int_{-\infty}^{\infty} \frac{1 + t^2}{t - z} \, d\rho(t).
$$

Hölder’s inequality now gives

$$
|\psi(z)|^2 \leq \int_{-\infty}^{\infty} \frac{1 + t^2}{|t - z|^2} \, d\rho(t) \cdot \int_{-\infty}^{\infty} (1 + t^2) \, d\rho(t)
= \frac{1}{\Im z} \int_{-\infty}^{\infty} \Im \left( \frac{1 + t^2}{t - z} \right) \, d\rho(t) \cdot \int_{-\infty}^{\infty} (1 + t^2) \, d\rho(t)
= \frac{\Im \psi(z)}{\Im z} \cdot \int_{-\infty}^{\infty} (1 + t^2) \, d\rho(t).
$$

Therefore

$$
D(\varphi) = \sup_{z \in \mathbb{H}} \frac{\Im z}{\Im \varphi(z)} = \sup_{z \in \mathbb{H}} \frac{\Im z |\psi(z)|^2}{\Im \psi(z)} \leq \int_{-\infty}^{\infty} (1 + t^2) \, d\rho(t) < \infty,
$$

from statement (b). Hence $S^h_\varphi$ is bounded on $L^1(\mathbb{R})$ by Theorem 5.1, with $\|S^h_\varphi\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} (1 + t^2) \, d\rho(t)$ from (5.7) and (5.8). The proof is complete. □
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