Please proofread your author proof very carefully and return it to Providence within 1 week.

- Only one set of author proof is sent out for papers with multiple authors.
- Please update address(es) at the end of the paper with any current address(es) you want added. Affiliations already in the TeX file will not be removed.
- Gray highlighting that appears in your proof is for internal AMS purposes to flag the use of \label, \ref, and \cite, which we encourage authors to use for maximum linking in the online product.

Any items checked below are specific to your paper:

- Some or all of your figures were touched for technical reasons. Please check all figures very carefully.
- Figures will appear in black and white in the print version of the journal.
- Color figures have been technically optimized for black and white use in the print product but will appear in color online. Please check all figures very carefully.
- Update your Mathematics Subject Classifications using the 2010 scheme found at http://www.ams.org/msc.

Return only one set of proofs to

Lauren Foster
American Mathematical Society
Electronic Prepress Department
201 Charles Street
Providence, RI 02904

or you may mark your corrections clearly and legibly on your proofs, then scan and return in an email attachment to lkf@ams.org

Thank you for publishing with the American Mathematical Society.
GROUPS OF UNITARY COMPOSITION OPERATORS
ON HARDY-SMIRNOV SPACES

GAJATH GUNATILLAKE, MIRJANA JOVOVIC, AND WAYNE SMITH

(Communicated by Pamela B. Gorkin)

Abstract. Let Ω be an open simply connected proper subset of the complex plane. We identify, up to isomorphism, which groups are possible for the group of unitary composition operators of a Hardy-Smirnov space defined on Ω. We also study the relationship between the geometry of Ω and the corresponding group.

1. Introduction

Let Ω be an open simply connected proper subset of the complex plane. The Hardy-Smirnov space $H^2(\Omega)$ is the Hilbert space of functions $F$ analytic on Ω such that the integrals of $|F|^2$ over the images of the circles $|z| = r$, $0 < r < 1$, under a Riemann map from the unit disc $\mathbb{D}$ onto Ω are uniformly bounded; see §3, Chapter 10 or §7, p. 63. A different choice of Riemann map onto Ω results in the same space of functions. The classical Hardy space $H^2$ of analytic functions on the unit disc corresponds to the choice $\Omega = \mathbb{D}$.

If $f \in H^2(\Omega)$ and $\phi$ is an analytic self map of Ω, then the composition operator induced by $\phi$ on $H^2(\Omega)$, denoted by $C_\phi$, is the linear operator defined by

$$C_\phi(f) = f \circ \phi.$$ Such operators on $H^2(\Omega)$ are studied in §7. Also, composition operators on a Hardy space of a half-plane are studied in §5, §6. It is elementary that the collection of all unitary operators on a Hilbert space forms a group with group operation the composition of operators. It is not difficult to show that the set of unitary composition operators on $H^2(\Omega)$ also forms a group, which we denote by $\mathcal{U}_\Omega$ (see Lemma 2.3).

In the case that $\Omega = \mathbb{D}$, it is known that unitary composition operators on $H^2$ are precisely those induced by rotations of $\mathbb{D}$, i.e. by the maps $\phi_\lambda(z) = \lambda z$, where $|\lambda| = 1$; see, for example, §4, Theorem 6]. Moreover since $C_{\phi_\lambda} C_{\phi_\mu} = C_{\phi_{\lambda\mu}}$, $\mathcal{U}_\mathbb{D}$ is isomorphic to the unit circle with ordinary multiplication of complex numbers.

Our goal in this paper is to show how the geometry of Ω determines $\mathcal{U}_\Omega$ and to identify, up to isomorphism, what groups are possible. In Section 4 it is shown that if $\mathcal{U}_\Omega$ is of finite order $n$, then Ω has $n$-fold symmetry and $\mathcal{U}_\Omega$ is isomorphic to $\mathbb{Z}_n$. The case that $\mathcal{U}_\Omega$ is an infinite group is studied in Section 5.

Received by the editors July 19, 2013 and, in revised form, December 16, 2013.

2010 Mathematics Subject Classification. Primary 47B33; Secondary 30H10.

Key words and phrases. Composition operator, unitary operator, Hardy-Smirnov space.

The first author would like to thank the University of Hawaii at Manoa for its generosity in hosting him during the collaboration.

© American Mathematical Society
we show that when $\mathcal{U}_\Omega$ is infinite, it is isomorphic to one of just five groups. Our results about the geometry of $\Omega$ leading to these groups are summarized in Theorem \[5.11\]. In particular, we show that $\mathcal{U}_\Omega$ is isomorphic to the unit circle with usual multiplication if and only if $\Omega$ is a disc.

2. Background material

Throughout this paper $\Omega$ will represent an open simply connected proper subset of the complex plane. Let $\gamma$ be a Riemann map that takes $\mathbb{D}$ onto $\Omega$. For $0 < r < 1$ let $\Gamma_r$ be the curve in $\Omega$ defined by $\Gamma_r = \gamma(\{|z| = r\})$. The set of functions analytic on $\Omega$ for which

$$\sup_{0 < r < 1} \int_{\Gamma_r} |f(w)|^2 |dw| < \infty$$

is a Hardy-Smirnov space on $\Omega$. We denote this space by $H^2(\Omega)$. The functional $\|\cdot\|_\Omega$ defined on $H^2(\Omega)$ by

$$\|f\|_\Omega = \left( \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_{\Gamma_r} |f(w)|^2 |dw| \right) \right)^{1/2}$$

is a norm on $H^2(\Omega)$ (see [7, p. 63]).

As mentioned above, $H^2(\mathbb{D})$ is the classical Hardy space $H^2$. For an arbitrary $\Omega$ it turns out that $H^2(\Omega)$ is isometrically isomorphic to $H^2$.

Theorem A. Suppose $f$ is holomorphic on $\Omega$. Then $f \in H^2(\Omega)$ if and only if $(f \circ \gamma)(\gamma')^{1/2} \in H^2$. The map $V$ given by $V(f) = (f \circ \gamma)(\gamma')^{1/2}$ is a linear isometry from $H^2(\Omega)$ onto $H^2$.

For a proof see [7, p. 63] or [8, p. 169]. Using $V$, we can define a function $\langle \cdot, \cdot \rangle_\Omega : H^2(\Omega) \times H^2(\Omega) \to \mathbb{C}$ by

$$\langle f, g \rangle_\Omega = \langle V(f), V(g) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $H^2$. Since $V$ is an isometry it follows that $\langle f, f \rangle_\Omega = \|V(f)\|^2_{H^2} = \|f\|^2_{H^2}$. Thus $H^2(\Omega)$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_\Omega$.

Suppose $C_\phi$ is a composition operator on $H^2(\Omega)$. If $g \in H^2$ and $z \in \mathbb{D}$, then

$$V C_\phi V^{-1}(g)(z) = \left( \frac{\gamma'(z)}{\gamma'(\varphi(z))} \right)^{1/2} g(\varphi(z)),$$

where $\varphi = \gamma^{-1} \circ \phi \circ \gamma$. Let $\psi(z) = (\gamma'(z)/\gamma'(\varphi(z)))^{1/2}$. Then $V C_\phi V^{-1}(g) = C_{\psi, \varphi}(g)$, where $C_{\psi, \varphi}(g) = \psi \cdot g \circ \varphi$. Such an operator is called a weighted composition operator. Moreover, $C_{\phi}^{-1}$ on $H^2(\Omega)$ is isometrically similar to the weighted composition operator $C_{\psi, \varphi}^{-1}$ on $H^2$ (see [8, p. 66]).

Lemma 2.1. If $C_\phi$ is invertible, then $\phi$ is an automorphism of $\Omega$ and $C_\phi^{-1} = C_{\phi^{-1}}$.

Proof. Clearly, $C_\phi$ is invertible if and only if $C_{\psi, \varphi} = V C_\phi V^{-1}$ is invertible on $H^2$, where $\varphi = \gamma^{-1} \circ \phi \circ \gamma$ and $\psi(z) = (\gamma'(z)/\gamma'(\varphi(z)))^{1/2}$. From Theorem 2.0.1 of [3] it follows that $C_{\psi, \varphi}$ is invertible if and only if $\varphi$ is an automorphism of $\mathbb{D}$ and $\psi$ and $1/\psi$ are in $H^\infty$. Moreover $C_{\psi, \varphi}^{-1} = C_{1/\psi, \varphi^{-1}}$. It follows from $\varphi = \gamma^{-1} \circ \phi \circ \gamma$ that $\phi$ is an automorphism of $\Omega$. If $g \in H^2(\Omega)$, then a direct computation yields

$$V^{-1} C_{\psi, \varphi}^{-1} V(g) = g \circ \phi^{-1}.$$ 

Thus $C_{\phi^{-1}}$ is a bounded operator on $H^2(\Omega)$ and $C_{\phi^{-1}}^{-1} = C_{\phi^{-1}}$. \qed
Interestingly, for certain $\Omega$ it is possible for an automorphism of $\Omega$ to induce a bounded composition operator that is not invertible \[8\].

**Lemma 2.2.** Suppose that $U$ is a bounded operator on $H^2(\Omega)$. Then $U$ is unitary if and only if $VUV^{-1}$ is unitary on $H^2$.

**Proof.** Let $T = VUV^{-1}$. Since $V$ is an isometry it is clear that $T$ is an isometry if and only if $U$ is an isometry. Also, $T$ is invertible if and only if $U$ is invertible. □

**Lemma 2.3.** The set of all unitary composition operators of $H^2(\Omega)$ is a group under the composition of operators.

**Proof.** Since $C^{-1}_\phi \circ f = C_{f^{-1}} \circ \phi$ and $C_f \circ C_g = C_{g \circ f}$ the result follows. □

**Notation**
- The group of unitary composition operators of $H^2(\Omega)$ is denoted by $U_\Omega$.
- The group of the unit circle with usual multiplication is denoted by $(\mathbb{T}, \cdot)$.
- The group of real numbers with usual addition is denoted by $(\mathbb{R}, +)$.
- The group of integers with usual addition is denoted by $(\mathbb{Z}, +)$.
- $\{1, -1\} \rtimes \mathbb{R}$ is used to denote the semidirect product of the multiplicative group $\{1, -1\}$ and $(\mathbb{R}, +)$ with group operation
  $$(a_1, t_1) * (a_2, t_2) = (a_1a_2, a_2t_1 + t_2)$$
  where $a_1, a_2 \in \{1, -1\}$ and $t_1, t_2 \in \mathbb{R}$.
- $\{1, -1\} \times \mathbb{Z}$ denotes the subgroup of $\{1, -1\} \times \mathbb{R}$ where the second coordinate is in $\mathbb{Z}$.
- Suppose that $f$ is a self map of the set $\Omega$. If $n \in \mathbb{N}$, then $f_n$ is the composition of $f$ with itself $n$ times. If $f$ is invertible and $m$ is a negative integer, then $f_m = f_{|m|}^{-1}$. Also $f_0$ denotes the identity map.

We leave it to the reader as an easy exercise to show that each of the five groups $(\mathbb{T}, \cdot), (\mathbb{R}, +), (\mathbb{Z}, +), \{1, -1\} \times \mathbb{R}$ and $\{1, -1\} \times \mathbb{Z}$ belongs to a different isomorphism class.

### 3. Unitary Operators

We begin our work by classifying those symbols $\phi$ which induce unitary composition operators on $H^2(\Omega)$.

**Definition 3.1.** Let $\mathcal{UA}_\Omega = \{ \phi : C_\phi$ is unitary on $H^2(\Omega) \}$.

**Theorem 3.2.** Suppose that $\phi$ is an analytic self map of $\Omega$. Then $\phi \in \mathcal{UA}_\Omega$ if and only if there exist $\mu, c \in \mathbb{C}, |\mu| = 1$, such that $\phi(w) = \mu w + c$ is an automorphism of $\Omega$.

**Proof.** First suppose that $\phi(w) = \mu w + c, |\mu| = 1, c \in \mathbb{C}$, is an automorphism of $\Omega$. Let $\varphi = \gamma^{-1} \circ \phi \circ \gamma$. Then

$$\phi \circ \gamma = \gamma \circ \varphi.$$

Therefore

$$\mu \gamma(z) + c = \gamma(\varphi(z))$$

for $z \in \mathbb{D}$. Now it follows that

$$\mu \gamma'(z) = \gamma'(\varphi(z)) \cdot \varphi'(z).$$
Since \( \phi \) is an automorphism of \( \Omega \), \( \varphi \) is an automorphism of \( \mathbb{D} \). Let \( \beta = \varphi^{-1}(0) \). Then \( \varphi(z) = \kappa(z - \beta)/(1 - \beta z) \) for some \( |\kappa| = 1 \). Hence

\[
\frac{\gamma'(z)}{\gamma'(\varphi(z))} = \frac{\mu K_{\beta}}{1 - |\beta|^2}
\]

If \( \psi(z) = \left( \frac{\gamma'(z)}{\gamma'(\varphi(z))} \right)^{1/2} \), then \( \psi = \lambda \frac{K_{\beta}}{\|K_{\beta}\|} \), where \( |\lambda| = 1 \) and \( K_{\beta}(z) = 1/(1 - \beta z) \) is the reproducing kernel for \( H^2 \); i.e. if \( f \in H^2 \), then \( \langle f, K_{\beta} \rangle = f(\beta) \). Thus \( C_{\psi,\varphi} \) is a unitary operator on \( H^2 \) [3] Theorem 6], and by Lemma 2.2 \( C_{\phi} \) is unitary on \( H^2(\Omega) \).

Now suppose that \( C_{\phi} \) is unitary on \( H^2(\Omega) \). Then \( C_{\psi,\varphi} \) is unitary on \( H^2 \), where

\[
\psi(z) = \frac{\gamma'(z)}{\gamma'(\varphi(z))}
\]

and \( \varphi = \gamma^{-1} \circ \phi \circ \gamma \). If \( C_{\phi} \) is invertible, then \( \phi \) is an automorphism of \( \Omega \); hence \( \varphi \) is an automorphism of \( \mathbb{D} \). If \( \beta = \varphi^{-1}(0) \), then \( \varphi(z) = \zeta(z - \beta)/(1 - \beta z) \) for some unimodular \( \zeta \). Since \( C_{\psi,\varphi} \) is unitary, \( \psi = \lambda K_{\beta}/\|K_{\beta}\| \) for some unimodular \( \lambda \) [3] Theorem 6]. Therefore

\[
\frac{\gamma'(z)}{\gamma'(\varphi(z))} = \lambda^2 \frac{1 - |\beta|^2}{(1 - \beta z)^2}.
\]

But \( \zeta \varphi'(z) = (1 - |\beta|^2)/(1 - \beta z)^2 \), and thus

\[
\gamma'(z) = \lambda^2 \zeta \gamma'(\varphi(z)) \varphi'(z).
\]

Therefore \( \gamma'(z) = \lambda^2 \zeta \gamma(\varphi(z))' \), and by integrating both sides

\[
\gamma(z) = \lambda^2 \zeta \gamma(\varphi(z)) + k.
\]

Since \( \gamma \circ \varphi = \phi \circ \gamma \), we have

\[
\gamma(z) = \lambda^2 \zeta \phi(\gamma(z)) + k.
\]

The desired result follows by letting \( w = \gamma(z) \). \( \square \)

4. Finite Order

**Lemma 4.1.** Suppose that \( \mathcal{U}_\Omega \) is a finite group. Further, suppose that \( \phi(w) = \mu w + c \) and \( \psi(w) = \mu w + d \). If \( \phi, \psi \in \mathcal{U}_A \), then \( c = d \).

**Proof.** It is readily seen that

\[
\phi^{-1}(w) = \mu^{-1} w - c \mu^{-1}.
\]

If \( \varphi = \psi \circ \phi^{-1} \), it follows that \( \varphi \in \mathcal{U}_A \Omega \) since \( C_{\phi^{-1}} \circ C_{\psi} \in \mathcal{U}_\Omega \). Now

\[
\varphi(w) = \mu (\mu^{-1} w - c \mu^{-1}) + d = w - c + d.
\]

It is easy to see that \( \varphi_m(w) = w + m(d - c) \) for any positive integer \( m \). If \( d - c \neq 0 \), then \( \{ \varphi_m : m = 1, 2, \cdots \} \) is infinite and thus \( G = \{ C_{\varphi_m} : m = 1, 2, \cdots \} \) is also infinite. But \( G \subseteq \mathcal{U}_\Omega \), and hence \( d - c = 0 \). \( \square \)

Suppose that \( n > 1 \) is an integer. The domain \( \Omega \) is said to have \( n \)-fold symmetry about the point \( p \) if \( e^{2\pi i/n}(\Omega - p) = \Omega - p \).

**Theorem 4.2.** If \( \mathcal{U}_\Omega \) has finite order \( n > 1 \), then \( \Omega \) has \( n \)-fold symmetry and \( \mathcal{U}_\Omega \) is isomorphic to \( \mathbb{Z}_n \).
Before giving the proof, we note that $n$-fold symmetry of $\Omega$ does not imply that $U_1$ has finite order. A simple example is when $\Omega$ is the unit disc $D$, which has $n$-fold symmetry for every integer $n$. As previously noted, $U_\otimes \cong (\mathbb{T}, \cdot)$. Another example is when $\Omega$ is a strip which has 2-fold symmetry, but no higher symmetry. See Proposition 5.8 below.

Proof. If $C_\phi \in U_2$, then it follows from Theorem 5.2 that $\phi(w) = \mu w + c$, $|\mu| = 1$. Since the order of $U_2$ is $n$, $C_\phi^n$ is the identity. But $C_\phi^n = C_{\phi_n}$ and $\phi_n(w) = \mu^n w + (1 + \mu + \cdots + \mu^{n-1})c$. Thus $\mu^n w + (1 + \mu + \cdots + \mu^{n-1})c = w$, which gives $\mu^n = 1$. Since $\phi$ is an automorphism, it follows that $\Omega$ has $n$-fold symmetry about $c/(1 - \mu)$, the fixed point of $\phi$.

Let $G_n$ denote the group \{\e^{2\pi ik/n} : k = 0, 1, \ldots, n - 1\} under usual complex number multiplication. Define $\Pi : U_2 \to G_n$ by

$$\Pi(C_\phi) = \phi'(0).$$

A straightforward computation shows that

$$\Pi(C_{\phi_1} \circ C_{\phi_2}) = \mu_1 \mu_2 = \Pi(C_{\phi_1})\Pi(C_{\phi_2}).$$

Thus $\Pi$ is a group homomorphism. From Lemma 4.1 it follows that $\Pi$ is one-to-one. Since $U_2$ and $G_n$ contain the same number of elements, $\Pi$ is onto as well. Hence $\Pi$ is an isomorphism. Clearly $G_n$ is isomorphic to $\mathbb{Z}_n$, and the result follows.

Example 4.3. Let $n > 1$ be an integer. Let $\Omega$ be the interior of the regular $n$-sided polygon whose vertices are $1, e^{2\pi i/n}, e^{2\pi i2/n}, \ldots, e^{2\pi i(n-1)/n}$. Then $U_2$ is isomorphic to $\mathbb{Z}_n$.

Proof. It is easy to see that $f(w) = e^{2\pi i\theta} w + c$ is an automorphism of $\Omega$ if and only if $\theta = m/n, m \in \mathbb{Z}$ and $c = 0$. The desired result now follows from Theorem 3.2.

The following example gives a domain $\Omega$ with $U_2$ the trivial group, showing that order one is possible.

Example 4.4. Let $\Omega = \{x + iy, x > 0, y > 0\}$. Then $U_2$ is the trivial group.

It is easily seen that $f(w) = e^{2\pi i\theta} w + c$ is an automorphism of $\Omega$ if and only if $f(w) = w$. Thus from Theorem 5.2 it follows that $U_2$ is the trivial group.

5. Infinite Order

In this section we work under the assumption that $U_2$ is of infinite order.

Lemma 5.1. If $f(w) = w + \alpha$ and $g(w) = w + \beta$ are distinct automorphisms of $\Omega$, then the line through $\alpha$ and $\beta$ contains the origin.

Proof. Assume, contrary to the statement, that $\alpha$ and $\beta$ do not lie on the same line through the origin. Note that this implies $\alpha \beta \neq 0$. Next, observe that $f$ and $g$ are automorphisms of $\Omega$ if and only if $\hat{f}(z) = z + 1$ and $\hat{g}(z) = z + \beta/\alpha$ are automorphisms of $\alpha^{-1}\Omega$. Thus we may assume that $\alpha = 1$ and $\beta \notin \mathbb{R}$. We further assume that $\Im \beta > 0$; if $\Im \beta < 0$ the proof will be similar, except that the orientation of the curve $\Gamma_N$ below will be reversed.

Let $w_0 \in \Omega$, so $f(w_0) = w_0 + 1 \in \Omega$ and $g(w_0) = w_0 + \beta \in \Omega$ as well. Since $\Omega$ is path connected, there are smooth curves $\gamma$ from $w_0$ to $w_0 + 1$ and $\eta$ from $w_0$ to $w_0 + \beta$ that lie in $\Omega$. Now fix

$$R > \max\{\text{diam}(\gamma), \text{diam}(\eta)\},$$

$$\frac{\text{diam}(\gamma)}{\text{dist}(\gamma, \partial \Omega)} = \frac{\text{diam}(\eta)}{\text{dist}(\eta, \partial \Omega)},$$

and let $\Gamma_N$ be the union of the curves $\gamma$ and $\eta$.
and choose an integer $N$ large enough so that the disc $B(0, 2R)$ with center 0 and radius $2R$ is contained in the interior of the parallelogram $P_N$ with vertices $w_0 \pm N \pm N\beta$.

Next recall from the end of §2 the definition of the iterates $\{f_n\}_{n\in\mathbb{Z}}$ and $\{g_n\}_{n\in\mathbb{Z}}$, and set

$$\sigma = \sum_{n=-N}^{N-1} f_n(\gamma) \quad \text{and} \quad \lambda = \sum_{n=-N}^{N-1} g_n(\eta).$$

Then the curve

$$\Gamma_N = g_{-N}(\sigma) + f_N(\lambda) - g_N(\sigma) - f_{-N}(\lambda) = \sum_{j=1}^{4} \Gamma_{Nj}$$

is a closed curve in $\Omega$ that passes through the vertices of $P_N$.

Now let $\zeta \in B(0, R)$. Note that since $B(0, 2R)$ is in the interior of $P_N$, it follows from (5.1) that each $\Gamma_{Nj}$ is contained in the complement of a half-plane that contains $B(0, R)$. Thus the increment of the argument of $z - \zeta$ over $\Gamma_{Nj}$ satisfies

$$0 < \Delta \arg(z - \zeta, \Gamma_{Nj}) < \pi.$$ 

Therefore $0 < \Delta \arg(z - \zeta, \Gamma_N) < 4\pi$, and hence $\Delta \arg(z - \zeta, \Gamma_N) = 2\pi$ since $\Gamma_N$ is a closed curve. Thus the winding number of $\Gamma_N$ about each point $\zeta \in B(0, R)$ is 1, and it follows that $B(0, R) \subset \Omega$. Since the only assumption on $R$ was (5.1), this contradicts the standing assumption that $\Omega \neq \mathbb{C}$ and completes the proof. \qed

**Lemma 5.2.** Suppose that $f(w) = \mu w + c$ and $g(w) = \mu w + d$ are both automorphisms of $\Omega$, where $\mu$ is unimodular. If $c \neq d$, then $\mu$ is 1 or $-1$.

**Proof.** Let $w \in \Omega$. Then

$$h(w) = f \circ g^{-1}(w) = w + c - d.$$ 

Let

$$p(w) = f \circ h \circ f^{-1}(w) = w + \mu(c - d).$$

Then $p$ and $h$ are automorphisms of $\Omega$. By Lemma 5.1, $c - d$ and $\mu(c - d)$ lie on the same line through the origin. Hence $\mu^2 = 1$. \qed

We saw in Theorem 3.2 that $f \in \mathcal{U}\mathcal{A}_\Omega$ has the form $f(w) = \mu w + c$, and so extends to be an entire function. We will always consider $f$ extended in this way, so that even if $0 \notin \Omega$ we can write $f(0) = c$ and $f'(0) = \mu$. This convention is used in the next definition and subsequently.

**Definition 5.3.** Let

$$R_\Omega = \{f'(0) : f \in \mathcal{U}\mathcal{A}_\Omega\}.$$ 

Also, $R_\Omega = \{\mu : f(w) = \mu w + c, f \text{ is an automorphism of } \Omega, |\mu| = 1\}$. Since the identity map is always in $\mathcal{U}\mathcal{A}_\Omega$ it is clear that $1 \in R_\Omega$. If $\lambda, \mu \in R_\Omega$, then there are $f, g \in \mathcal{U}\mathcal{A}_\Omega$ so that $f(w) = \lambda w + c$ and $g(w) = \mu w + d$. Since both $f \circ g, f^{-1} \in \mathcal{U}\mathcal{A}_\Omega$, it follows that $\lambda \mu \in R_\Omega$ and $\lambda^{-1} \in R_\Omega$. Hence $R_\Omega$ is a subgroup of $(\mathbb{T}, \cdot)$.

The set $R_\Omega$ can be infinite or finite. We leave the proof of the next lemma as an easy exercise.

**Lemma 5.4.** If $R_\Omega$ is infinite, then it is dense in the unit circle.

**Lemma 5.5.** Suppose that $\mathcal{U}_\Omega$ is infinite. Then
(1) The following are equivalent:
(a) $R_{\Omega}$ is infinite.
(b) $\Omega$ is a disc.
(c) $R_{\Omega} = \mathbb{T}$.

(2) If $\Omega$ is not a disc, then $R_{\Omega} \subseteq \{1, -1\}$.

Proof. (a) $\implies$ (b): Suppose that $R_{\Omega}$ is infinite. Choose $\mu_1, \mu_2 \in R_{\Omega}$ such that $(\mu_1 \mu_2)^2 \neq 1$. Then there are $f, g \in \mathcal{UA}_{\Omega}$ so that $f(w) = \mu_1 w + c_1$ and $g(w) = \mu_2 w + c_2$. Now
\[ f \circ g(w) = \mu_1 \mu_2 w + (\mu_1 c_2 + c_1) \text{ and } g \circ f(w) = \mu_2 \mu_1 w + (\mu_2 c_1 + c_2). \]
Since $f \circ g \circ f \in \mathcal{UA}_{\Omega}$ and $\mu_1 \mu_2 \neq \pm 1$, it follows from Lemma 5.2 that $\mu_1 c_2 + c_1 = \mu_2 c_1 + c_2$. Thus
\[ (1 - \mu_2)c_1 = (1 - \mu_1)c_2. \]

Since $R_{\Omega}$ is dense in $\mathbb{T}$ we can choose a dense sequence $\{\mu_n\}_{n=1}^{\infty}$ from $R_{\Omega}$ such that $(\mu_i)^2 \neq 1$ and $(\mu_i \mu_{i+1})^2 \neq 1$. For each $\mu_i$ there is an automorphism $f(w) = \mu_i w + c_i$. Since $(\mu_i \mu_{i+1})^2 \neq 1$, it follows from (5.3) that $c_i/(1 - \mu_i) = c_{i+1}/(1 - \mu_{i+1})$ for $i = 2, 3, \ldots$. Let $p = c_1/(1 - \mu_1)$ and
\[ E = \Omega - p. \]
We first show that $\mu_j E \subseteq E$. If $v \in E$, then $v = w - p$ for some $w \in \Omega$. Now
\[ \mu_j v + p = \mu_j w - \frac{\mu_j c_j}{1 - \mu_j} + \frac{c_j}{1 - \mu_j} = \mu_j w + c_j. \]
Since $\mu_j v \in E$ if and only if $\mu_j v + p \in \Omega$ it is readily seen that $\mu_j v \in E$.

Let $z \in \mathbb{C} \setminus E$. Since $z + p \in \mathbb{C} \setminus \Omega$ and $h(w) = \mu_j w + c_j$ is an automorphism of $\Omega$ that extends to $\mathbb{C}$, it follows that $\mu_j(z + p) + c_j \in \mathbb{C} \setminus \Omega$. Since $\mu_j z = \mu_j(z + p) + c_j - p$ it follows that $\mu_j z \in \mathbb{C} \setminus E$. Thus
\[ \{\mu_j z | j = 1, 2, \cdots \} \subseteq \mathbb{C} \setminus E. \]
Note that $\mathbb{C} \setminus E$ is closed. The closure of $\{\mu_j z | j = 1, 2, \cdots \}$ is a circle centered at 0 which is contained in $\mathbb{C} \setminus E$. Also, $E$ is open and connected; hence $E$ is a disc centered at the origin. Thus $\Omega$ is a disc centered at $p$.

(b) $\implies$ (c): If $\Omega$ is a disc with center $p$, then $w \to e^{i\theta}(w - p) + p$ is an automorphism of $\Omega$ for all real $\theta$. Thus $R_{\Omega} = \mathbb{T}$.

The proof of (c) $\implies$ (a) is trivial.

Proof of (2): Since $\Omega$ is not a disc, from part (1) it follows that $R_{\Omega}$ is finite. If $|\lambda| = 1, \lambda^2 \neq 1$, then from Lemma 5.2 there is at most one $c'$ such that $k(w) = \lambda w + c'$ is an automorphism of $\Omega$. But $R_{\Omega}$ is infinite; thus there are infinitely many automorphisms of the form $f(w) = w + d$ or $g(w) = -w + k$. Let $\mu \in R_{\Omega}$. Then there is $h(w) = \mu w + c$ so that $h \in \mathcal{UA}_{\Omega}$. Since
\[ h \circ f(w) = \mu w + (\mu d + c) \text{ and } h \circ g(w) = -\mu w + (\mu k + c) \]
it follows from Lemma 5.2 that $\mu^2 = 1$. \qed

Part (1) of the previous lemma gives a complete description of $\Omega$ when $R_{\Omega}$ is infinite. Next we turn to the case that $R_{\Omega}$ is finite, so $R_{\Omega} \subseteq \{1, -1\}$ by Lemma 5.2.
From Lemma 5.3 there is a unimodular constant \( \lambda \) such that if \( f(w) = w + c \) is in \( \mathcal{UA}_\Omega \), then \( \overline{\lambda c} \) is real. Define

\[
S_\Omega = \{ t \in \mathbb{R} : f_t(w) = w + t\lambda \text{ is in } \mathcal{UA}_\Omega \}.
\]

Suppose that \( \inf \{ t > 0 : t \in S_\Omega \} \) is real. Define

\[
\Omega = \{ w \in \mathbb{C} : f_t(w) \text{ is in } \mathcal{UA}_\Omega \text{ for all } t \in S_\Omega \}.
\]

Then there are infinitely many of the latter, take \( \bigcup \Omega \) and is parallel to \( \Phi \).

Let \( \Omega \) be as in the definition of \( S_\Omega \). Let \( L_\lambda \) be the line \( \{ \lambda t : t \in \mathbb{R} \} \). If \( w \in \Omega \), then there is a \( \delta > 0 \) such that \( \Omega \) contains the line segment

\[
L_{w,\delta} = \{ w + \lambda t : -\delta < t < \delta \}.
\]

But \( \phi_n(L_{w,\delta}) = L_{w,\delta} + n\lambda t_w \). Thus \( L_w = \bigcup_{n \in \mathbb{Z}} \phi_n(L_{w,\delta}) \) is a line that goes through \( w \) and is parallel to \( L_\lambda \).

Since \( w \) is an arbitrary point in \( \Omega \), it follows that \( \bigcup_{w \in \Omega} L_w \subseteq \Omega \). Clearly \( \Omega \subseteq \bigcup_{w \in \Omega} L_w \); thus

\[
\Omega. \bigcup_{w \in \Omega} L_w.
\]

Since \( \Omega \neq \mathbb{C} \) and is connected, it must be a strip or a half-plane. Thus \( S_\Omega = \mathbb{R} \). Clearly \( -1 \in R_\Omega \) implies that \( \Omega \) is not a half-plane, and so must be a strip. Finally, the 2-fold symmetry of a strip implies that \( -1 \in R_\Omega \) if \( \Omega \) is a strip.

Proof of (2): First suppose that \( c_\Omega > 0 \). Since \( (S_\Omega, +) \) is a subgroup of \( (\mathbb{R}, +) \), if \( c_\Omega > 0 \), then \( S_\Omega \) has no point of accumulation and hence \( c_\Omega \in S_\Omega \). It follows that \( S_\Omega = \{ n \cdot c_\Omega : n \in \mathbb{Z} \} \). Sufficiency easily follows since \( S_\Omega \) is a nontrivial group. \( \square \)

If \( \Omega \) is the upper half-plane, then \( R_\Omega = \{ 1 \} \). Next we will look at such regions.

**Proposition 5.7.** Suppose that \( R_\Omega = \{ 1 \} \). Then

1. The following are equivalent:
   a. \( c_\Omega = 0 \).
   b. \( \Omega \) is a half-plane.
   c. \( \mathcal{U}_\Omega \) is isomorphic to \( (\mathbb{R}, +) \).
(2) If \( c_Ω > 0 \), then \( U_Ω \) is isomorphic to \((\mathbb{Z}, +)\).

**Proof.** (a) \( \implies \) (b): It follows from (1) of Lemma 5.6 that \( Ω \) is a half-plane.

(b) \( \implies \) (c): Since \( Ω \) is a half-plane there is a \( λ ∈ \mathbb{C} \) so that the boundary of \( Ω \) is parallel to the line \( l_λ = \{ λt : t ∈ \mathbb{R} \} \). If \( t ∈ \mathbb{R} \), and \( φ(w) = w + tλ \), then \( φ ∈ U_A Ω \).

Define \( Π : U_Ω → (\mathbb{R}, +) \) by

\[ Π(C_φ) = λ^{-1}φ(0). \]

It is readily seen that \( Π \) is a bijection. If \( f(w) = w + λt_1, g(w) = w + λt_2 \), then \( g ∘ f(w) = w + λ(t_1 + t_2) \). Since \( C_f ∘ C_g = C_{g ∘ f} \) we get

\[ Π(C_f ∘ C_g) = t_1 + t_2 = Π(C_f) + Π(C_g). \]

Thus \( U_Ω \) is isomorphic to \((\mathbb{R}, +)\).

(c) \( \implies \) (a): Suppose that \( U_Ω \cong (\mathbb{R}, +) \). Clearly \( c_Ω ≥ 0 \), and if \( c_Ω > 0 \), then part (2) yields that \( U_Ω \cong (\mathbb{Z}, +) \). Thus \( c_Ω = 0 \).

Proof of (2): This is clear from (2) of Lemma 5.6. □

If \( Ω = \{ z : -1 < \text{Im}(z) < 1 \} \), then \( R_Ω = \{ 1, -1 \} \). Next we look at such regions.

**Proposition 5.8.** Suppose that \( R_Ω = \{ 1, -1 \} \). Then

(1) The following are equivalent:

(a) \( c_Ω = 0 \).

(b) \( Ω \) is a strip.

(c) \( U_Ω \) is isomorphic to \( \{ 1, -1 \} × \mathbb{R} \).

(2) If \( c_Ω > 0 \), then \( U_Ω \) is isomorphic to \( \{ 1, -1 \} × \mathbb{Z} \).

**Proof.** (a) \( \implies \) (b): The result follows from (1) of Lemma 5.6.

(b) \( \implies \) (c): Since \( Ω \) is a strip there is some \( λ ∈ \mathbb{C} \) so that \( Ω \) is parallel to the line \( l_λ = \{ λt : t ∈ \mathbb{R} \} \). Let \( z_0 \) be the point with minimum modulus such that

\[ -Ω + z_0 = Ω. \]

Therefore \( s(w) = -w + z_0 ∈ U_A Ω \). For any \( r(w) = -w + d ∈ U_A Ω \) we have \( r ∘ s(w) = w + d - z_0 ∈ U_A Ω \); thus \( d - z_0 = λt \) for some \( t ∈ \mathbb{R} \). Hence if \( f ∈ U_A Ω \) and \( f''(0) = -1 \), then \( f(0) = z_0 + λt \). Thus, \( C_φ ∈ U_Ω \) if and only if \( φ(w) = w + λt \) or \( φ(w) = -w + z_0 + λt, t ∈ \mathbb{R} \).

Let \( Π : U_Ω → \{ 1, -1 \} × \mathbb{R} \) be defined by

\[ Π(C_φ) = \left( φ'(0), λ^{-1} \left( φ(0) - \frac{z_0}{2} (1 - φ'(0)) \right) \right). \]

It is easy to see that \( Π \) is a bijection. If \( φ ∈ U_A Ω \), then it can be written in the form \( φ(w) = μw + z_0(1 - μ)/2 + λt \). Suppose that \( h, s ∈ U_A Ω \) with \( h(w) = μ_1w + z_0(1 - μ_1)/2 + λt_1 \) and \( s(w) = μ_2w + z_0(1 - μ_2)/2 + λt_2 \). A straightforward computation yields that

\[ Π(C_h ∘ C_s) = (μ_2μ_1, μ_2t_1 + t_2) = Π(C_h) ∗ Π(C_s). \]

Thus, \( U_Ω \cong \{ 1, -1 \} × \mathbb{R} \).

(c) \( \implies \) (a): Suppose that \( U_Ω \cong \{ 1, -1 \} × \mathbb{R} \). Clearly \( c_Ω ≥ 0 \), and if \( c_Ω > 0 \), then part (2) yields that \( U_Ω \cong \{ 1, -1 \} × \mathbb{Z} \). Thus \( c_Ω = 0 \).

Proof of (2): If \( c_Ω > 0 \), then from (2) of Lemma 5.6 we see that \( S_Ω = \{ n ∗ c_Ω : n ∈ \mathbb{Z} \} \). Recall that there is a unimodular \( λ \) such that if \( f ∈ U_A Ω \) and \( f''(0) = 1 \), then \( f(w) = w + λt \) for some \( t ∈ S_Ω \). If \( g(w) = -w + c \) and \( h(w) = -w + d \) are automorphisms of \( Ω \), then \( g ∘ h(w) = w + c - d \) is also an automorphism of \( Ω \); hence \( c - d = n ∗ c_Ωλ \) for some \( n ∈ \mathbb{Z} \). Thus, there is a \( d_0 ∈ \mathbb{C} \) such that if
$h \in \mathcal{U}_\Omega$ and $h'(0) = -1$, then $h(w) = -w + d_0 + m \cdot c_\Omega \lambda$ for some $m \in \mathbb{Z}$. Let $\Pi : \mathcal{U}_\Omega \to \{1, -1\} \times \mathbb{Z}$ be defined by

$$
\Pi(C_\phi) = \left( \phi'(0), (c_\Omega \lambda)^{-1} \left( \phi(0) - \frac{d_0}{2} (1 - \phi'(0)) \right) \right).
$$

Then it is easy to see that $\Pi$ is an isomorphism. \hfill \square

**Proposition 5.9.** The group $(\mathbb{T}, \cdot)$ is isomorphic to $\mathcal{U}_\Omega$ if and only if $\Omega$ is a disc.

**Proof.** First suppose that $\Omega$ is a disc centered at the origin. It is readily seen that $f \in \mathcal{U}_A \Omega$ if and only if $f(w) = e^{2\pi i \theta} w$ for some $\theta \in \mathbb{R}$ (Theorem 3.2). The map $\Pi : \mathcal{U}_\Omega \to (\mathbb{T}, \cdot)$ defined by

$$
\Pi(C_\phi) = \phi'(0)
$$

shows that $\mathcal{U}_\Omega$ is isomorphic to $(\mathbb{T}, \cdot)$. If $\Omega$ is a disc centered at $p$, then the same proof works with the domain $\Omega - p$.

Next we prove the other direction. Suppose that $\mathcal{U}_\Omega$ is isomorphic to $(\mathbb{T}, \cdot)$. From Proposition 5.7 and Proposition 5.8 it follows that $R_\Omega \not\subseteq \{1, -1\}$. But $\mathcal{U}_\Omega$ is infinite; hence the desired result follows from Lemma 5.5. \hfill \square

We summarize all the results in the theorem below.

**Theorem 5.10.** If $\mathcal{U}_\Omega$ is infinite, then it is isomorphic to one of the following groups:

1. $(\mathbb{T}, \cdot)$,
2. $(\mathbb{R}, +)$,
3. $(\mathbb{Z}, +)$,
4. $\{1, -1\} \times \mathbb{R}$,
5. $\{1, -1\} \times \mathbb{Z}$.

**Proof.** If $R_\Omega$ is infinite, then from Lemma 5.5 and Proposition 5.9 it follows that $\mathcal{U}_\Omega \cong (\mathbb{T}, \cdot)$.

If $R_\Omega$ is not infinite, then from Lemma 5.5 it is contained in $\{1, -1\}$. Notice that 1 is always contained in $R_\Omega$. Thus $R_\Omega = \{1\}$ or $R_\Omega = \{1, -1\}$. If $R_\Omega = \{1\}$, then Proposition 5.7 yields that $\mathcal{U}_\Omega$ is isomorphic to either $(\mathbb{R}, +)$ or $(\mathbb{Z}, +)$. If $R_\Omega = \{1, -1\}$, then from Proposition 5.8 it follows that $\mathcal{U}_\Omega \cong \{1, -1\} \times \mathbb{R}$ or $\mathcal{U}_\Omega \cong \{1, -1\} \times \mathbb{Z}$. \hfill \square

In some of these cases we can give a precise description of $\Omega$.

**Theorem 5.11.** If $\mathcal{U}_\Omega$ is infinite, then the following are true.

1. $\mathcal{U}_\Omega \cong (\mathbb{T}, \cdot)$ if and only if $\Omega$ is a disc.
2. $\mathcal{U}_\Omega \cong (\mathbb{R}, +)$ if and only if $\Omega$ is a half-plane.
3. $\mathcal{U}_\Omega \cong \{1, -1\} \times \mathbb{R}$ if and only if $\Omega$ is a strip.

**Proof.** Proposition 5.9 yields (1); (2) follows from Proposition 5.7, and Proposition 5.8 gives (3). \hfill \square

However, many different $\Omega$ can lead to $(\mathbb{Z}, +)$ or $\{1, -1\} \times \mathbb{Z}$. We close the paper with a few examples.

**Example 5.12.** Let

$$
\Omega = \{x + iy : y > \sin(x)\}.
$$

Then $\mathcal{U}_\Omega \cong (\mathbb{Z}, +)$. 
It is easy to see that $f \in \mathcal{UA}_\Omega$ if and only if $f(w) = w + 2\pi n, n \in \mathbb{Z}$. Thus $R_\Omega = \{1\}$, and the result follows from Proposition 5.7.

Example 5.13. Let
$$\Omega = \{ x + iy : 1 + \sin(x) > y > \sin(x) \}.$$ Then $\mathcal{U}_\Omega \cong \{1, -1\} \rtimes \mathbb{Z}$.

It is readily seen that $f \in \mathcal{UA}_\Omega$ if and only if $f(w) = w + 2\pi n, n \in \mathbb{Z}$, or $f(w) = -w + i + (2n)\pi, n \in \mathbb{Z}$. Hence $R_\Omega = \{1, -1\}$, and Proposition 5.8 yields the desired result.

Example 5.14. For $k \in \mathbb{Z}$ define $N_k^+ = \{2k + yi : 1/2 \leq y < 1\}$ and $N_k^- = \{(2k+1) + yi : -1 < y \leq -1/2\}$. Let
$$\Omega = \{ z : -1 < \text{Im}(z) < 1 \} \setminus \bigcup_{k \in \mathbb{Z}} (N_k^+ \cup N_k^-).$$

Then $\mathcal{U}_\Omega \cong \{1, -1\} \rtimes \mathbb{Z}$.

Clearly $f \in \mathcal{UA}_\Omega$ if and only if $f(w) = w + 2n$ or $f(w) = -w + 1 + 2n, n \in \mathbb{Z}$. Thus $R_\Omega = \{1, -1\}$, and the result follows from Proposition 5.8.

References