COMPOSITION OPERATORS ON SMALL SPACES

BOO RIM CHOE, HYUNGWOON KOO, AND WAYNE SMITH

ABSTRACT. We show that if a small holomorphic Sobolev space on the unit disk is not just small but very small, then a trivial necessary condition is also sufficient for a composition operator to be bounded. A similar result for holomorphic Lipschitz spaces is also obtained. These results may be viewed as boundedness analogues of Shapiro's theorem concerning compact composition operators on small spaces. We also prove the converse of Shapiro's theorem if the symbol function is already contained in the space under consideration. In the course of the proofs we characterize the bounded composition operators on the Zygmund class. Also, as a by-product of our arguments, we show that small holomorphic Sobolev spaces are algebras.

1. INTRODUCTION

Throughout the paper φ denotes a nonconstant holomorphic self-map of the unit disk $\mathbf{D}$ in the complex plane $\mathbf{C}$. Associated with φ is the composition operator C_φ defined by

$$C_\varphi f = f \circ \varphi$$

for $f \in H(\mathbf{D})$, the space of all functions holomorphic on $\mathbf{D}$. It is clear that C_φ takes $H(\mathbf{D})$ into itself.

The main subject in the study of composition operators is to describe operator theoretic properties of C_φ in terms of function theoretic properties of φ . A recent book [3] of Cowen and MacCluer is a good reference for the theory of composition operators. In the present paper we are mainly concerned about composition operators on holomorphic Sobolev spaces $A_{\alpha,s}^p$ defined in terms of fractional derivatives. We will define these spaces and briefly review their basic properties in Section 3. Also, we refer to Section 3 for definitions of the well-known weighted Bergman spaces A_α^p and the Hardy spaces $H^p = A_{-1}^p$.

By the equivalence property (3.3) below, we have $A_{\alpha,s}^p \approx A_{\alpha-sp}^p$ for $sp < \alpha + 1$. It is a well-known consequence of Littlewood's Subordination Principle that every composition operator is bounded on A_α^p for every $p > 0$ and $\alpha \geq -1$. So, every composition operator is bounded on $A_{\alpha,s}^p$ whenever $sp < \alpha + 1$. This boundedness property extends to parameters $p \geq 2$, $s > 0$ and $\alpha > -1$ with $sp = \alpha + 1$. Moreover, for parameters other than those mentioned above, some composition operators fail to be bounded on $A_{\alpha,s}^p$. These are recent results of the authors [2].

For the boundedness of composition operators and other related properties, extensive study in special cases such as weighted Hardy spaces $A_{\alpha,s}^2$ and Dirichlet-type

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spaces $A_{\alpha,1}^p$ has already been done; see [3]. However, characterizing when a composition operator is bounded on general $A_{\alpha,s}^p$ is wide open. The authors [2] tried such a general approach and obtained various partial results concerning boundedness. Roughly speaking, parameters there are restricted at most to $sp < \alpha + 2 + p$.

A main motivation for the present paper is a result of J. H. Shapiro [12] (also, Theorem 4.5 of [3]) asserting that the condition $\|\varphi\|_\infty < 1$ is necessary for C_φ to be compact on a “suitably small” Banach space. In [12] four axioms are required for suitably small spaces; two are minimal axioms concerning norm naturality and nontriviality of the spaces. The other two axioms are those which control the size of the spaces. That is, suitably small spaces are small by the boundary regularity axiom (continuous extension property up to the boundary), but *not* “too small” by the automorphism-invariance axiom. See [12] or [3] for details.

We want to apply Shapiro’s result to the spaces $A_{\alpha,s}^p$, even when $0 < p < 1$ and $A_{\alpha,s}^p$ is not a Banach space. In this case, however, the functional $\|\cdot\|_{A_{\alpha,s}^p}^p$ is a p -norm on $A_{\alpha,s}^p$, meaning that it is subadditive and homogeneous of order p , and the induced metric is complete. Even though the result in [12] mentioned in the previous paragraph is stated for Banach spaces, it remains valid in the context of p -Banach spaces with the same proof. What needs to be observed is that the tools used in [12], such as the Closed Graph Theorem and the spectral radius formula, are valid in the p -Banach space setting; see [6, Satz 5.7], [14] and [11, Chapter IV]. We are grateful to Joel Shapiro for discussions regarding this material.

The spaces $A_{\alpha,s}^p$ in general satisfy all the axioms from [12] except the boundary regularity axiom; see the remark after (3.2) below for the automorphism-invariance. So, $A_{\alpha,s}^p$ is a suitably small space if and only if $A_{\alpha,s}^p$ is contained in $C(\mathbf{D})$. Based on the Sobolev-type embedding theorem (3.6) and the Privalov theorem (3.8) below, the parameter range that ensures boundary regularity is as follows:

$$\begin{aligned} \text{either} \quad & \bullet \quad sp > \alpha + 2; \\ \text{or} \quad & \bullet \quad sp = \alpha + 2, \quad 0 < p \leq 1. \end{aligned} \tag{1.1}$$

We write $(s, p, \alpha) \in \mathcal{R}$ when (1.1) is satisfied. Thus, when restricted to holomorphic Sobolev spaces, Shapiro’s result is as follows.

Theorem 1.1 (Shapiro). *Suppose $(s, p, \alpha) \in \mathcal{R}$. If C_φ is compact on $A_{\alpha,s}^p$, then $\|\varphi\|_\infty < 1$.*

Recall that when $sp > \alpha + 1$, some composition operators are not bounded on $A_{\alpha,s}^p$. The reason is very simple; the proof of Theorem 4.4 of [2] shows that there exists $\varphi \notin A_{\alpha,s}^p$. Moreover, our example in Section 4 shows that some $\varphi \in A_{\alpha,s}^p$ fail to induce a bounded composition operator on $A_{\alpha,s}^p$ even for $sp > \alpha + 2$. This demonstrates that some suitably small spaces are *too large* for φ in that space to induce a bounded composition operator.

Thus, in view of general philosophy underlying Theorem 1.1, one may expect (as we do) that if a small space X is not just suitably small but “very small”, then the trivial necessary condition $\varphi = C_\varphi z \in X$ would be also sufficient for C_φ to be bounded. Positive evidence for such an insight comes from a result of Zorboska [15], who worked on weighted Hardy spaces. Here, $H_s^2 = A_{-1,s}^2$.

Theorem 1.2 (Zorboska). *Assume $s > \frac{3}{2}$. Then C_φ is bounded on H_s^2 if and only if $\varphi \in H_s^2$.*

Note that $H_s^2 \subset \Lambda_{s-1/2}$ by (3.6) below. Here, Λ_a denotes a holomorphic Lipschitz space of order $a > 0$ described in Section 2. So, when $s > \frac{3}{2}$, the space H_s^2 is not just a suitably small space, but also has additional boundary smoothness of order at least 1, which suggests a plausible condition for very small spaces. The following result on such holomorphic Lipschitz spaces themselves provides another positive evidence.

Theorem 1.3. *Let $a > 1$. Then C_φ is bounded on Λ_a if and only if $\varphi \in \Lambda_a$.*

The embedding theorems (3.6) and (3.8) below indicate that the parameter range that ensures the boundary smoothness of order at least 1 is precisely $(s-1, p, \alpha) \in \mathcal{R}$. Thus, we are eventually led to the following result for holomorphic Sobolev spaces. Our proof actually gives a bit more; see Theorem 3.3.

Theorem 1.4. *Suppose $(s-1, p, \alpha) \in \mathcal{R}$ and assume that s is a positive integer if $\alpha = -1$. Then C_φ is bounded on $A_{\alpha,s}^p$ if and only if $\varphi \in A_{\alpha,s}^p$.*

It would be quite surprising if Theorem 1.4 would fail to hold for some fractional s in case $\alpha = -1$. But, we do not have a proof. The difficulty is that we cannot use (3.3) below (unless $p = 2$) to get an isomorphism with a space of functions defined in terms of full derivatives.

An example will be presented in Section 4 showing that in Theorem 1.4 the restriction on the parameters is sharp. However, the less restrictive condition $sp > \alpha + 2$ in Theorem 1.1 suggests that a version of Theorem 1.4 with “bounded” replaced by “compact” may be valid for this greater range of parameters. This, as well as the Lipschitz analogue, turns out to be correct.

Theorem 1.5. *Let $a > 0$. Then C_φ is compact on Λ_a if and only if $\varphi \in \Lambda_a$ and $\|\varphi\|_\infty < 1$.*

Theorem 1.6. *Suppose $(s, p, \alpha) \in \mathcal{R}$ and assume that s is a positive integer if $\alpha = -1$. Then C_φ is compact on $A_{\alpha,s}^p$ if and only if $\varphi \in A_{\alpha,s}^p$ and $\|\varphi\|_\infty < 1$.*

As in the case of the boundedness result, this again recovers a result of Zorboska [15] for weighted Hardy spaces H_s^2 . The sufficiency of the above theorem, which is our contribution, is actually true for $sp \geq \alpha + p$; see Theorem 3.7.

Section 2 is devoted to the study of holomorphic Lipschitz spaces. We prove Theorem 1.3 and Theorem 1.5. In the course of the proof, we obtain the characterization of bounded composition operators on the Zygmund class (Theorem 2.2). Section 3 is devoted to the study of holomorphic Sobolev spaces. We prove strengthened versions of Theorem 1.4 and Theorem 1.6; see Theorem 3.3 and Theorem 3.7 respectively. As a by-product of our proofs, we show that holomorphic Sobolev spaces with parameters in $\mathcal{R}$ are algebras; see Theorem 3.5. Finally, in Section 4, we construct various examples showing that our results are sharp in the sense that the range of parameters cannot be improved.

2. LIPSCHITZ SPACES

We recall well known holomorphic Lipschitz spaces. For $0 < \epsilon < 1$, we let Λ_ϵ denote the space of all $f \in H(\mathbf{D}) \cap C(\overline{\mathbf{D}})$ such that

$$\|f\|_{\Lambda_\epsilon} = \sup \frac{|f(z) - f(w)|}{|z - w|^\epsilon} < \infty \quad (2.1)$$

where the supremum is taken over all $z, w \in \partial D$, $z \neq w$. For $\epsilon = 1$, we let Λ_1 denote the space of all $f \in H(\mathbf{D}) \cap C(\overline{\mathbf{D}})$ such that

$$\|f\|_{\Lambda_1} = \sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty$$

where the supremum is taken over all $e^{i\theta} \in \partial D$ and $h > 0$. Of course, these are semi-norms since they do not distinguish between functions differing by a constant ($0 < \epsilon < 1$) or by a linear polynomial ($\epsilon = 1$). As usual, they may be made into norms by adding $|f(0)|$ or $|f(0)| + |f'(0)|$. All of our results are valid for the normed spaces, though we will continue to work with the semi-norms to simplify our exposition.

Comparing with the Lipschitz space Lip_1 corresponding to $\epsilon = 1$ in (2.1), we have $\text{Lip}_1 \subset \Lambda_1$. The notion of holomorphic Lipschitz spaces extends to arbitrary order in a standard way. For a positive integer n and $0 < \epsilon \leq 1$, we let $\Lambda_{n+\epsilon}$ denote the space of all $f \in H(\mathbf{D}) \cap C^n(\overline{\mathbf{D}})$ such that $f^{(n)} \in \Lambda_\epsilon$; the semi-norm of $f \in \Lambda_{n+\epsilon}$ is given by

$$\|f\|_{\Lambda_{n+\epsilon}} = \|f^{(n)}\|_{\Lambda_\epsilon}. \quad (2.2)$$

There are two classical results characterizing these spaces in terms of the growth rate of derivatives. The first one, due to Hardy and Littlewood, is the characterization of Λ_ϵ , $0 < \epsilon < 1$:

$$\|f\|_{\Lambda_\epsilon} \approx \sup_{z \in \mathbf{D}} (1 - |z|)^{1-\epsilon} |f'(z)| \quad (2.3)$$

for $f \in H(\mathbf{D})$. See Theorem 5.1 of [4] or Theorem 4.1 of [3]. Here and below we use the abbreviated notation $A \lesssim B$ to mean $A \leq CB$ for some inessential constant $C > 0$, and $A \approx B$ if $A \lesssim B \lesssim A$.

The second one, due to Zygmund, is the characterization of Λ_1 :

$$\|f\|_{\Lambda_1} \approx \sup_{z \in \mathbf{D}} |f''(z)|(1 - |z|) \quad (2.4)$$

for $f \in H(\mathbf{D})$. See Theorem 5.3 in [4]. For this reason, the space Λ_1 is often called the Zygmund class. Note that we have by (2.4)

$$|f'(z) - f'(0)| \lesssim \|f\|_{\Lambda_1} \log \left(\frac{1}{1 - |z|} \right), \quad z \in \mathbf{D} \quad (2.5)$$

The following result is due to K. Madigan [10] (also, see Theorem 4.9 of [3]).

Lemma 2.1. *Let $0 < \epsilon < 1$. Then C_φ is bounded on Λ_ϵ if and only if*

$$\sup_{z \in \mathbf{D}} |\varphi'(z)| \left(\frac{1 - |z|}{1 - |\varphi(z)|} \right)^{1-\epsilon} < \infty. \quad (2.6)$$

Also, it is well known that C_φ is bounded on Lip_1 if and only if $\varphi' \in H^\infty$. However, we are not aware of any published characterization of when a composition operator is bounded on the Zygmund class. Here, we provide a characterization.

For that purpose, consider the following three conditions on φ :

$$\sup_{z \in \mathbf{D}} |\varphi'(z)| \left[\log \left(\frac{2}{1-|z|} \right) \right]^{-1} \log \left(\frac{2}{1-|\varphi(z)|} \right) < \infty, \quad (2.7)$$

$$\sup_{z \in \mathbf{D}} \frac{|\varphi'(z)|^2(1-|z|)}{1-|\varphi(z)|} < \infty, \quad (2.8)$$

$$\sup_{z \in \mathbf{D}} |\varphi''(z)|(1-|z|) \log \left(\frac{2}{1-|\varphi(z)|} \right) < \infty. \quad (2.9)$$

We then have the following characterization.

Theorem 2.2. *The following are equivalent:*

- (i) C_φ is bounded on Λ_1 .
- (ii) (2.7) and (2.9) hold.
- (iii) (2.8) and (2.9) hold.

Proof. We first prove the implication (i) $\implies$ (ii). So, assume that C_φ is bounded on Λ_1 . We then necessarily have $\varphi \in \Lambda_1$ and thus

$$|\varphi''(z)|(1-|z|) \lesssim 1, \quad |\varphi'(z)| \lesssim \log \left(\frac{2}{1-|z|} \right), \quad z \in \mathbf{D} \quad (2.10)$$

by (2.4) and (2.5). We now use test functions f_ζ defined by

$$f_\zeta''(z) = \frac{\zeta}{1-\zeta z}, \quad f_\zeta(0) = f_\zeta'(0) = 0, \quad \zeta \in \partial\mathbf{D}.$$

It is clear that $\|f_\zeta\|_{\Lambda_1} \lesssim 1$ for any ζ by (2.4). Thus, the boundedness of C_φ on Λ_1 yields $\|f_\zeta(\varphi)\|_{\Lambda_1} \lesssim 1$. Note $f_\zeta'(z) = \log(1-\zeta z)^{-1}$. It follows from (2.5) and (2.4) that

$$|[f_\zeta(\varphi)]''(z)| \lesssim \frac{1}{1-|z|}, \quad |[f_\zeta(\varphi)]'(z)| \lesssim \log \left(\frac{2}{1-|z|} \right) \quad (2.11)$$

for $z \in \mathbf{D}$ and $\zeta \in \partial\mathbf{D}$. Now, fix z and take ζ such that $\zeta\varphi(z) = |\varphi(z)|$. First, we obtain from the second inequality in (2.11) that

$$|\varphi'(z)| \log \left(\frac{1}{1-|\varphi(z)|} \right) \lesssim \log \left(\frac{2}{1-|z|} \right), \quad (2.12)$$

which, together with (2.10), yields the estimate (2.7). Note that we have

$$\frac{1-|z|^2}{1-|\varphi(z)|^2} \leq \frac{1-|\varphi(0)|^2}{|1-\overline{\varphi(0)}\varphi(z)|^2} \leq \frac{1+|\varphi(0)|}{1-|\varphi(0)|} \quad (2.13)$$

by the Schwarz-Pick lemma. Therefore, it follows from (2.12) and (2.13) that

$$\frac{|\varphi'(z)|^2(1-|z|)}{1-|\varphi(z)|} \lesssim \left\{ \log \left(\frac{2}{1-|z|} \right) / \log \left(\frac{1}{1-|\varphi(z)|} \right) \right\}^2 \frac{1-|z|}{1-|\varphi(z)|} \lesssim 1; \quad (2.14)$$

the second estimate used the elementary facts that $h(x) = x[\log(1/x)]^2$ is an increasing and concave function of x positive and small, so that $h(x) \leq h(Cx) \leq Ch(x)$ for $C \geq 1$. We can now deduce from the first inequality in (2.11) and (2.14) that

$$|\varphi''(z)|(1-|z|) \log \left(\frac{1}{1-|\varphi(z)|} \right) \lesssim \frac{|\varphi'(z)|^2(1-|z|)}{1-|\varphi(z)|} + 1 \lesssim 1.$$

This, together with (2.10), completes the estimate (2.9).

The implication (ii) $\implies$ (iii) was proved above in showing that (2.12) implies (2.14).

Finally, we prove the implication (iii) $\implies$ (i). So, assume that the conditions (2.8) and (2.9) hold. Let $f \in \Lambda_1$. We wish to prove $f(\varphi) \in \Lambda_1$. To this end, it is sufficient to show that

$$\sup_{z \in \mathbf{D}} |[f(\varphi)]''(z)|(1 - |z|) < \infty. \quad (2.15)$$

Note that $\varphi \in \Lambda_1$ by (iii) and (2.4). Thus, we may assume $f'(0) = 0$. Therefore, we have by (2.4) and (2.5)

$$\begin{aligned} |[f(\varphi)]''(z)| &\leq |f''(\varphi(z))||\varphi'(z)|^2 + |f'(\varphi(z))||\varphi''(z)| \\ &\lesssim \|f\|_{\Lambda_1} \left\{ \frac{|\varphi'(z)|^2}{1 - |\varphi(z)|} + |\varphi''(z)| \log \left(\frac{1}{1 - |\varphi(z)|} \right) \right\} \end{aligned}$$

for $z \in \mathbf{D}$. This, together with (2.8) and (2.9), implies (2.15). The proof is complete. $\square$

In what follows, given a function space $X \subset H(\mathbf{D})$, we let $\mathcal{M}(X)$ denote the pointwise multiplier algebra of X . More explicitly, we let

$$\mathcal{M}(X) = \{g \in H(\mathbf{D}) : fg \in X \text{ for any } f \in X\}.$$

The pointwise multiplier algebras of holomorphic Lipschitz spaces are easily identified. Note that the containment $\mathcal{M}(\Lambda_a) \subset \Lambda_a$ is clear. The converse containment $\Lambda_a \subset \mathcal{M}(\Lambda_a)$ is also immediate from (2.3) and (2.4). In summary we have

$$\mathcal{M}(\Lambda_a) = \Lambda_a, \quad a > 0. \quad (2.16)$$

We are now ready to prove Theorem 1.3 and Theorem 1.5. Here, we restate them for convenience.

Theorem 2.3. *For $a > 1$, C_φ is bounded on Λ_a if and only if $\varphi \in \Lambda_a$. For $a > 0$, C_φ is compact on Λ_a if and only if $\varphi \in \Lambda_a$ and $\|\varphi\|_\infty < 1$.*

Proof. First, we consider the boundedness. Since $\varphi = C_\varphi z$, the necessity is clear. We now prove the sufficiency. So, assume $\varphi \in \Lambda_a$. First consider the case $1 < a \leq 2$. Note that $\varphi' \in C(\overline{\mathbf{D}})$. By Lemma 2.1, we see that C_φ is bounded on Λ_{a-1} for $1 < a < 2$. For $a = 2$ we have $\varphi' \in \Lambda_1$ and thus φ'' is at most of logarithmic growth as in (2.5). So, we see from (2.13) that

$$|\varphi''(z)|(1 - |z|) \log \left(\frac{2}{1 - |\varphi(z)|} \right) \lesssim (1 - |z|) \left\{ \log \left(\frac{2}{1 - |z|} \right) \right\}^2 \lesssim 1$$

for $z \in \mathbf{D}$. Meanwhile, since $\varphi' \in C(\overline{\mathbf{D}})$, we have

$$\frac{|\varphi'(z)|^2(1 - |z|)}{1 - |\varphi(z)|} \leq \|\varphi'\|_\infty \cdot 2, \quad z \in \mathbf{D} \quad (2.17)$$

by the Schwarz-Pick lemma. It follows from Theorem 2.2 that C_φ is bounded on Λ_1 for $a = 2$. Accordingly, by (2.16), we obtain

$$\begin{aligned} \| [f(\varphi)] \|_{\Lambda_a} &= \| [f'(\varphi)] \varphi' \|_{\Lambda_{a-1}} \\ &\lesssim \| [f'(\varphi)] \|_{\Lambda_{a-1}} \| \varphi' \|_{\Lambda_{a-1}} \\ &\lesssim \| f' \|_{\Lambda_{a-1}} \| \varphi' \|_{\Lambda_{a-1}} \\ &= \| f \|_{\Lambda_a} \| \varphi \|_{\Lambda_a} \end{aligned}$$

for each $1 < a \leq 2$.

Now, we proceed by induction on the range of a . So, suppose the theorem is true for $n < a \leq n+1$ for some integer $n \geq 1$. Consider the next range $n+1 < a \leq n+2$. By induction hypothesis, C_φ is bounded on Λ_{a-1} . Meanwhile, since $\varphi \in \Lambda_a$, we have $\varphi' \in \Lambda_{a-1} = \mathcal{M}(\Lambda_{a-1})$ by (2.16). Thus, proceeding as above, we conclude that C_φ is bounded on Λ_a for $n+1 < a \leq n+2$, as desired.

Next, we consider the compactness. As usual, the necessity of $\varphi \in \Lambda_a$ comes from the observation that $\varphi = C_\varphi z$, while the necessity of $\|\varphi\|_\infty < 1$ is clear since Λ_a is a “suitably small” Banach space satisfying the four axioms required for Shapiro’s theorem discussed in the Introduction. For the sufficiency, assume $\varphi \in \Lambda_a$ and $\|\varphi\|_\infty < 1$. We first consider the case $0 < a < 1$. Let $\{f_n\}$ be a bounded sequence in Λ_a . To see that C_φ is compact on Λ_a , we must show that $\{C_\varphi f_n\}$ has a convergent subsequence. Now $\{f_n\}$ is a normal family on $\mathbf{D}$, so consider a normal limit f of some subsequence, which for convenience we continue to denote by $\{f_n\}$. It is easy to check that $f \in \Lambda_a$, so by subtracting f from each term we may assume that $\{f_n\}$ converges to 0 uniformly on compact subsets of $\mathbf{D}$. Then

$$|(f_n \circ \varphi)'(z)|(1 - |z|)^{a-1} = |f'_n(\varphi(z))| \cdot |\varphi'(z)|(1 - |z|)^{a-1} \lesssim |f'_n(\varphi(z))| \cdot \|\varphi\|_{\Lambda_a},$$

which converges to 0 uniformly on $\mathbf{D}$ as $n \rightarrow \infty$, by the uniform convergence to 0 of $\{f'_n\}$ on $\overline{\varphi(\mathbf{D})}$. Hence $\{C_\varphi f_n\}$ converges to 0 in Λ_a , which completes the proof in the case that $0 < a < 1$. The proof in the case $a = 1$ is similar, using (2.4) in place of (2.3). We omit the details.

As in the proof of boundedness, we now proceed by induction on the range of a . So suppose the theorem has been proved for $a \leq n$, where $n \geq 1$ is an integer, and consider a with $n < a \leq n+1$. Notice that the chain rule $(f \circ \varphi)' = f'(\varphi) \cdot \varphi'$ can be written

$$DC_\varphi = M_{\varphi'} C_\varphi D,$$

where D and $M_{\varphi'}$ represent the linear operators of differentiation and multiplication by φ' . Note that C_φ is bounded on Λ_a by what we’ve just proved above. From (2.2) we see that D is an isometry of Λ_a onto Λ_{a-1} , and so C_φ is compact on Λ_a if and only if $DC_\varphi = M_{\varphi'} C_\varphi D : \Lambda_a \rightarrow \Lambda_{a-1}$ is compact. In the term $M_{\varphi'} C_\varphi D$, $M_{\varphi'}$ is acting on the space Λ_{a-1} and is bounded by (2.16), while C_φ is acting on Λ_{a-1} also, and so is compact by the induction hypothesis. Hence $M_{\varphi'} C_\varphi D$ is compact, and the proof is complete. $\square$

Note that the proof of Theorem 2.3 shows that $\varphi \in \Lambda_a$ with $1 < a \leq 2$ induces a bounded composition operator on Λ_{a-1} . In fact we have the following corollary as a consequence of Theorem 2.3.

Corollary 2.4. *If C_φ is bounded (compact) on Λ_a for some $a > 0$, then C_φ is bounded (compact) on Λ_b for $0 < b < a$.*

Proof. The result for compactness is immediate from Theorem 2.3. For boundedness, the case $0 < a < 1$ is clear by (2.13) and Lemma 2.1. Next, assume that C_φ is bounded on Λ_1 . Then we have (2.7), or equivalently (2.12), by Theorem 2.2(ii). So, given $0 < \epsilon < 1$, a similar argument to the estimate in (2.14) leads to (2.6) and thus C_φ is bounded on Λ_ϵ . Finally, let $a > 1$ and assume that C_φ is bounded on Λ_a . If $1 < b < a$, the result is immediate from Theorem 1.3. Next we consider the case $b = 1$, and note that Theorem 1.3 allows us to assume $1 < a < 2$. Since $\varphi' \in \Lambda_{a-1}$, we have by (2.3) and (2.13)

$$|\varphi''(z)|(1 - |z|) \log \left(\frac{2}{1 - |\varphi(z)|} \right) \lesssim (1 - |z|)^{a-1} \log \left(\frac{2}{1 - |z|} \right) \lesssim 1.$$

This, together with (2.17) and Theorem 2.2, shows C_φ is bounded on Λ_1 . It remains to consider $0 < b < 1 < a$. We just showed that this implies C_φ is bounded on Λ_1 , which was shown earlier in the proof to imply that C_φ is bounded on Λ_b whenever $0 < b < 1$. This completes the proof. $\square$

3. SOBOLEV SPACES

Following [1], we define the s -fractional derivative for $f \in H(\mathbf{D})$ by

$$\mathcal{R}^s f(z) = \sum_{n=0}^{\infty} (1+n)^s a_n z^n, \quad z \in \mathbf{D}$$

provided each a_n is the n -th Taylor coefficient of f at the origin.

In order to introduce holomorphic Sobolev spaces of fractional order, we first recall well-known function spaces. Given $\alpha > -1$, let dA_α denote the weighted measure $dA_\alpha(z) = (1 - |z|^2)^\alpha dA(z)$ where dA is area measure on $\mathbf{D}$. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A_α^p is the space of all $f \in H(\mathbf{D})$ for which

$$\|f\|_{A_\alpha^p}^p = \int_{\mathbf{D}} |f(z)|^p dA_\alpha(z) < \infty.$$

Also, the Hardy space H^p is the space of all $f \in H(\mathbf{D})$ for which

$$\|f\|_{H^p}^p = \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

We often use the following notation to allow unified statements:

$$A_{-1}^p = H^p.$$

This notation is justified by the weak-star convergence of $(\alpha+1)dA_\alpha(z)/\pi$ to $d\theta/2\pi$ as $\alpha \rightarrow -1$.

Now, for $p > 0$, $s \geq 0$ and $\alpha \geq -1$, the holomorphic Sobolev space $A_{\alpha,s}^p$ is defined to be the space of all $f \in H(\mathbf{D})$ for which $\mathcal{R}^s f \in A_\alpha^p$. Again, we often write $H_s^p = A_{-1,s}^p$ as mentioned earlier. The norm of $f \in A_{\alpha,s}^p$ is given by

$$\|f\|_{A_{\alpha,s}^p} = \|\mathcal{R}^s f\|_{A_\alpha^p}.$$

Of course, we are abusing the term “norm”, since $\|\cdot\|_{A_{\alpha,s}^p}$ does not satisfy the triangle inequality for $0 < p < 1$, but in this case $(f, g) \mapsto \|f - g\|_{A_{\alpha,s}^p}^p$ defines a translation-invariant metric which turns $A_{\alpha,s}^p$ into a complete topological vector

space. In case of integral order, the Sobolev norm can be naturally computed in terms of full derivatives. More explicitly, we have

$$\|f\|_{A_{\alpha,s}^p} \approx \sum_{k=0}^s \|f^{(k)}\|_{A_{\alpha}^p} \approx \sum_{k=0}^{s-1} |f^{(k)}(0)| + \|f^{(s)}\|_{A_{\alpha}^p} \quad (3.1)$$

for each positive integer s ; see Theorem 5.3 of [1] for the first equivalence and Proposition 2.2 of [2] for the second one. The case $p = 2$ is much simpler to handle due to the Hilbert space structure. Namely, the estimates can be done by means of Taylor coefficients. Note that, given $f \in H(\mathbf{D})$ with Taylor series expansion $f(z) = \sum a_n z^n$, we have

$$\begin{aligned} \|f\|_{A_{\alpha,s}^2}^2 &= \sum (n+1)^{2s} |a_n|^2 \int_{\mathbf{D}} |z|^{2n} (1-|z|^2)^{\alpha} dA(z) \\ &= \pi \sum (n+1)^{2s} |a_n|^2 \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} \\ &\approx \sum (n+1)^{2s-\alpha-1} |a_n|^2 \end{aligned}$$

by Sterling's formula. Thus, we have

$$A_{\alpha,s}^2 \approx H_{s-(\alpha+1)/2}^2. \quad (3.2)$$

Here, we extend the notion of Hardy-Sobolev space H_s^2 to any real number s . So, as a corollary of Theorem 1.4, we recover Theorem 1.2 mentioned in the Introduction.

Remark. Bergman-Sobolev spaces can be identified with those spaces defined in terms of full derivatives by (3.3) below. So, it is clear from (3.1) that Bergman-Sobolev spaces of any order and Hardy-Sobolev spaces of integral order are automorphism invariant. For the remaining Hardy-Sobolev spaces of fractional order, one may use theories developed in [1] to prove the same automorphism invariance. Since we do not need such automorphism invariance in any significant way in this paper, we omit the details.

We briefly review some basic properties of holomorphic Sobolev spaces. So, let the parameters $p > 0$, $\alpha_j \geq -1$, $s_j \geq 0$ ($j = 1, 2$) be given. First, it is a general phenomenon that the order of differentiation and the weight can be canceled out as long as the weight remains bigger than -1 . More precisely, for $\alpha_1, \alpha_2 > -1$, we have the following equivalence property:

$$\alpha_1 - \alpha_2 = p(s_1 - s_2) \implies A_{\alpha_1, s_1}^p \approx A_{\alpha_2, s_2}^p. \quad (3.3)$$

That is, these spaces are isomorphic and have equivalent norms. This equivalence does not extend to Hardy-Sobolev spaces. However, we have an embedding property of the Littlewood-Paley-type:

$$\alpha_1 + 1 = p(s_1 - s_2), \quad p \leq 2 \implies A_{\alpha_1, s_1}^p \subset H_{s_2}^p; \quad (3.4)$$

the direction of this embedding is reversed for $p \geq 2$, but we do not need it here.

We now recall various embedding theorems which will be repeatedly used in our arguments. The first one is the embedding theorem between holomorphic Sobolev spaces:

$$\frac{\alpha_1 + 2}{p_1} - \frac{\alpha_2 + 2}{p_2} = s_1 - s_2, \quad p_1 < p_2 \implies A_{\alpha_1, s_1}^{p_1} \subset A_{\alpha_2, s_2}^{p_2} \quad (3.5)$$

and the next one is the holomorphic version of the well-known Sobolev embedding theorem:

$$sp > \alpha + 2 \implies A_{\alpha,s}^p \subset \Lambda_{s-(\alpha+2)/p}. \quad (3.6)$$

In the critical case we have

$$sp = \alpha + 2 \implies A_{\alpha,s}^p \subset \text{VMOA}; \quad (3.7)$$

see [5] for the definition and related facts about the space VMOA. All the embeddings above are continuous. We refer to Section 5 of [1] for details of all the equivalence and embedding properties mentioned above. There is yet another continuous embedding property we are interested in:

$$sp = \alpha + 2, \quad p \leq 1 \implies A_{\alpha,s}^p \subset H_1^1 \subset C(\overline{D}) \cap \mathcal{A} \quad (3.8)$$

where $\mathcal{A}$ denotes the class of functions absolutely continuous on ∂D . The first embedding holds by (3.5) and (3.4), while the second one is a classical theorem of Privalov (Theorem 3.11 of [4]).

We now proceed to the proof of Theorem 1.4, but first we need more notation. For a multi-index or m -tuple of nonnegative integers $J_m = (j_1, \dots, j_m)$, we let

$$\begin{aligned} |J_m| &= j_1 + j_2 + \dots + j_m \\ ||J_m|| &= j_1 + 2j_2 + \dots + mj_m \end{aligned}$$

and

$$g^{J_m} = [g']^{j_1} [g'']^{j_2} \dots [g^{(m)}]^{j_m}$$

for $g \in H(\mathbf{D})$. With this notation higher order derivatives of composite functions can be expanded as in the next lemma. The proof is an elementary induction and thus omitted. One can find various versions of this expansion formula with explicit coefficients in [8].

Lemma 3.1. *Given a positive integer s , there exist coefficients $c(s, J_m)$ such that*

$$[f(\varphi)]^{(s)} = \sum_{m=1}^s f^{(s-m+1)}(\varphi) \left(\sum_{\substack{||J_m||=s \\ |J_m|=s-m+1}} c(s, J_m) \varphi^{J_m} \right)$$

for functions $f \in H(\mathbf{D})$.

The following lemma provides the key estimate used in the proof of Theorem 1.4.

Lemma 3.2. *Suppose $(s-1, p, \alpha) \in \mathcal{R}$ and assume that $s \geq 2$ is an integer. Let $m \leq s-1$ be a positive integer and let φ be any holomorphic self-map of $\mathbf{D}$. Then there is a constant $C = C(\varphi, s, p, \alpha, m)$ such that*

$$||f(\varphi)g^{J_m}||_{A_{\alpha}^p} \leq C ||f||_{A_{\alpha,m-1}^p} ||g||_{A_{\alpha,s}^p}^{|J_m|}$$

for functions $f \in A_{\alpha,m-1}^p$, $g \in A_{\alpha,s}^p$ and multi-indices J_m with $|J_m| = ||J_m|| - m + 1$.

Proof. Fix $f \in A_{\alpha,m-1}^p$, $g \in A_{\alpha,s}^p$ and a multi-index J_m with $|J_m| = ||J_m|| - m + 1$. Since $(s-1, p, \alpha) \in \mathcal{R}$ by assumption, we have two cases to consider:

- (i) $(s-1)p > \alpha + 2$;
- (ii) $(s-1)p = \alpha + 2$ and $p \leq 1$.

The proof is somewhat long and thus divided into these two cases.

Case (i): Since $s - \frac{\alpha+2}{p} > 1$, there exists a unique integer $n \geq 1$ such that

$$n < s - \frac{\alpha+2}{p} \leq n+1. \quad (3.9)$$

Note that we have

$$A_{\alpha,s}^p \subset \Lambda_{s-(\alpha+2)/p} \subset C^n(\overline{\mathbf{D}}) \quad (3.10)$$

by (3.6). So, decompose $J_m = J_m^1 + J_m^2$ where $J_m^1 = (j_1, \dots, j_n, 0, \dots, 0)$ and $J_m^2 = J_m - J_m^1$. Then we have $g^{J_m^1} \in C(\overline{\mathbf{D}})$ by (3.10) and thus

$$\|g^{J_m^1}\|_\infty \lesssim \|g\|_{A_{\alpha,s}^p}^{|J_m^1|}.$$

Accordingly, it suffices to show

$$\|f(\varphi)g^{J_m^2}\|_{A_\alpha^p} \lesssim \|f\|_{A_{\alpha,m-1}^p} \|g\|_{A_{\alpha,s}^p}^{|J_m^2|}. \quad (3.11)$$

Since every composition operator is bounded on A_α^p , when $J_m^2 = 0$ this is a consequence of (3.1). So, assume $J_m^2 \neq 0$. Note that $m \geq n+1$. We introduce auxiliary exponents σ_k by

$$\frac{1}{\sigma_k} = j_k \left\{ 1 - \frac{p(s-k)}{\alpha+2} \right\}, \quad k = n+1, \dots, m. \quad (3.12)$$

From (3.9) we have $0 < \sigma_k \leq \infty$ for each $k \geq n+1$. Motivation for introducing these exponents is the embedding

$$1 \leq \sigma_k < \infty \implies A_{\alpha,s}^p \subset A_{\alpha,k}^{pj_k \sigma_k}, \quad (3.13)$$

which come from (3.5). Also, the embeddings are continuous. With regard to these, note that we have

$$\sigma_k = \infty \iff j_k = 0 \quad \text{or} \quad k = n+1 = s - \frac{\alpha+2}{p} \quad (3.14)$$

by (3.9). We choose a positive σ (possibly ∞) such that

$$\frac{1}{\sigma} = \sum_{k=n+1}^m \frac{1}{\sigma_k}.$$

We introduce one more auxiliary exponent λ by

$$\frac{1}{\lambda} = 1 - \frac{p(m-n)}{\alpha+2}.$$

Since $n+1 \leq m \leq s-1$, using (3.9) we see $1 < \lambda \leq \infty$. We also remark that

$$\lambda < \infty \implies A_{\alpha,m-n}^p \subset A_\alpha^{p\lambda} \quad (3.15)$$

$$\lambda = \infty \iff m = s-1 = n + \frac{\alpha+2}{p} \quad (3.16)$$

by (3.5) and (3.9). Next, put

$$a_n = 1 - \frac{p(s-n)}{\alpha+2},$$

and note that $a_n < 0$ by (3.9). Also, note that

$$\frac{1}{\sigma_k} = \frac{j_k}{s-n} \{a_n(s-k) + k - n\}.$$

It follows that

$$\begin{aligned}
(s-n) \left(1 - \frac{1}{\sigma}\right) &= |a_n| \sum_{k=n+1}^m j_k(s-k) + (s-n) - \sum_{k=n+1}^m j_k(k-n) \\
&= |a_n| \sum_{k=n+1}^m j_k(s-k) + s - \|J_m\| + \|J_m^1\| + n(|J_m^2| - 1)
\end{aligned} \tag{3.17}$$

Meanwhile, we have

$$\begin{aligned}
\frac{s-n}{\lambda} &= a_n(m-n) + (s-m) \\
&= a_n(m-n) + s - \|J_m\| + |J_m| - 1.
\end{aligned} \tag{3.18}$$

We now deduce from (3.17) and (3.18) that

$$\begin{aligned}
(s-n) \left(1 - \frac{1}{\sigma} - \frac{1}{\lambda}\right) &= |a_n| \left\{ \sum_{k=n+1}^m j_k(s-k) + (m-n) \right\} \\
&\quad + \|J_m^1\| - |J_m^1| + (n-1)(|J_m^2| - 1) \\
&\geq |a_n| \left\{ \sum_{k=n+1}^m j_k(s-k) + (m-n) \right\} \\
&> 0
\end{aligned}$$

where strict inequality holds, because $m-n \geq 1$. Accordingly,

$$\frac{1}{\sigma} + \frac{1}{\lambda} < 1$$

and, in particular, $\sigma > 1$. Let σ' be the conjugate index of σ . Note that $\sigma' < \lambda$. We separate the proof into two sub-cases for convenience: (a) $\sigma_{n+1} < \infty$ and (b) $\sigma_{n+1} = \infty$.

Sub-Case (a): In this case we have $\lambda \leq j_{n+1}\sigma_{n+1} < \infty$, because $m \leq s-1$ and $j_{n+1} \geq 1$. We also have $\sigma \leq \sigma_{n+1} < \infty$. Note that we have

$$\|f(\varphi)g^{J_m^2}\|_{A_\alpha^p} \leq \|f(\varphi)\|_{A_\alpha^{p\sigma'}} \|g^{J_m^2}\|_{A_\alpha^{p\sigma}} \tag{3.19}$$

by Hölder's inequality. Now, using (3.1) for the first inclusion and (3.15) for the second one, we have

$$A_{\alpha, m-1}^p \subset A_{\alpha, m-n}^p \subset A_\alpha^{p\lambda} \subset A_\alpha^{p\sigma'} \xrightarrow{C_\varphi} A_\alpha^{p\sigma'}$$

with all embeddings and C_φ continuous. This yields

$$\|f(\varphi)\|_{A_\alpha^{p\sigma'}} \lesssim \|f\|_{A_{\alpha, m-1}^p}. \tag{3.20}$$

Thus, in order to prove (3.11), it is sufficient to prove

$$\|g^{J_m^2}\|_{A_\alpha^{p\sigma}} \lesssim \|g\|_{A_{\alpha, s}^{p\sigma}}. \tag{3.21}$$

To see this, we first note that

$$\|g^{J_m^2}\|_{A_\alpha^{p\sigma}} \leq \prod_{\substack{\sigma_k < \infty \\ n+1 \leq k \leq m}} \left\| [g^{(k)}]^{j_k} \right\|_{A_\alpha^{p\sigma_k}}$$

by Hölder's inequality using $\sum_{k=n+1}^m \sigma/\sigma_k = 1$. On the other hand we have

$$\left\| [g^{(k)}]^{j_k} \right\|_{A_{\alpha}^{p\sigma_k}} = \|g^{(k)}\|_{A_{\alpha}^{pj_k\sigma_k}}^{j_k} \lesssim \|g\|_{A_{\alpha,k}^{pj_k\sigma_k}}^{j_k} \lesssim \|g\|_{A_{\alpha,s}^{p_{\alpha,s}^{j_k}}}^{j_k}$$

for each k with $n+1 \leq k \leq m$ and $\sigma_k < \infty$; the inequalities hold by (3.1) and (3.13). Recall that $\sigma_{n+1} < \infty$. Also, for $n+2 \leq k \leq m$, note that $\sigma_k < \infty$ if and only if $j_k \geq 1$ by (3.14). Thus, combining the above two observations, we have (3.21), as desired.

Sub-Case (b): In this case we have $\sigma_{n+1} = \infty$. This time we apply Hölder's inequality in a slightly different way to get that

$$\left\| f(\varphi)g^{J_m^2} \right\|_{A_{\alpha}^p} \leq \left\| f(\varphi)[g^{(n+1)}]^{j_{n+1}} \right\|_{A_{\alpha}^{p\sigma'}} \left\| g^{J_m^3} \right\|_{A_{\alpha}^{p\sigma}}$$

where $J_m^3 = (j_{n+2}, \dots, j_m)$. Since $\sum_{k=n+2}^m 1/\sigma_k = 1/\sigma$, we have

$$\left\| g^{J_m^3} \right\|_{A_{\alpha}^{p\sigma}} \lesssim \|g\|_{A_{\alpha,s}^{p_{\alpha,s}^{J_m^3}}}^{J_m^3}$$

for $\sigma < \infty$ by exactly the same argument used to prove (3.21). Moreover, it vacuously holds for $\sigma = \infty$, because in that case we have $J_m^3 = 0$ by (3.14). It remains to prove

$$\left\| f(\varphi)[g^{(n+1)}]^{j_{n+1}} \right\|_{A_{\alpha}^{p\sigma'}} \lesssim \|f\|_{A_{\alpha,m-1}^p} \|g\|_{A_{\alpha,s}^{p_{\alpha,s}^{j_{n+1}}}}^{j_{n+1}}. \quad (3.22)$$

If $\lambda < \infty$, then we have $j_{n+1} = 0$ by (3.14) and (3.16), and so estimate (3.22) was established in (3.20). So, assume $\lambda = \infty$. We then necessarily have by (3.16)

$$m = s - 1 = n + \frac{\alpha + 2}{p}$$

and so (3.6) and (3.7) show that

$$A_{\alpha,m-1}^p \subset \Lambda_{(s-2)-(\alpha+2)/p} = \Lambda_{n-1} \subset \text{VMOA}, \quad (3.23)$$

where $\Lambda_0 = \text{VMOA}$ (recall $n \geq 1$). Both embeddings above are also continuous. Since $s - \frac{\alpha+2}{p} = n+1$, we also have by (3.1) and (3.7)

$$g^{(n+1)} \in A_{\alpha,s-(n+1)}^p \subset \text{VMOA}. \quad (3.24)$$

Before proceeding, we recall the well-known continuous embeddings

$$\text{VMOA} \subset H^q \subset A_{\beta}^q$$

valid for any $0 < q < \infty$ and $\beta \geq -1$. So, using (3.23) we have

$$A_{\alpha,m-1}^p \subset \text{VMOA} \subset A_{\alpha}^{p\sigma'} \xrightarrow{C_{\varphi}} A_{\alpha}^{p\sigma'}$$

and thus obtain

$$\|f(\varphi)\|_{A_{\alpha}^{p\sigma'}} \lesssim \|f\|_{A_{\alpha,m-1}^p},$$

which is (3.22) for $j_{n+1} = 0$. For $j_{n+1} \geq 1$, we may use continuity of both

$$A_{\alpha,s-(n+1)}^p \subset \text{VMOA} \subset A_{\alpha}^{2p\sigma'j_{n+1}} \quad \text{and} \quad A_{\alpha,m-1}^p \subset \text{VMOA} \subset A_{\alpha}^{2p\sigma'} \xrightarrow{C_{\varphi}} A_{\alpha}^{2p\sigma'}$$

(the first one is given by (3.24)) to conclude from (3.24) and (3.1) that

$$\begin{aligned}
\left\| f(\varphi) [g^{(n+1)}]^{j_{n+1}} \right\|_{A_{\alpha}^{p\sigma'}} &\leq \|f(\varphi)\|_{A_{\alpha}^{2p\sigma'}} \left\| [g^{(n+1)}]^{j_{n+1}} \right\|_{A_{\alpha}^{2p\sigma'}} \\
&\lesssim \|f\|_{A_{\alpha}^{2p\sigma'}} \left\| g^{(n+1)} \right\|_{A_{\alpha}^{2p\sigma' j_{n+1}}}^{j_{n+1}} \\
&\lesssim \|f\|_{A_{\alpha, m-1}^p} \left\| g^{(n+1)} \right\|_{A_{\alpha, s-(n+1)}^p}^{j_{n+1}} \\
&\lesssim \|f\|_{A_{\alpha, m-1}^p} \|g\|_{A_{\alpha, s}^p}^{j_{n+1}},
\end{aligned}$$

as desired. This completes the proof of Sub-Case (b) and the proof of Case (i).

Case (ii): While the proof is much simpler, general scheme of the proof is the same. Since $s - \frac{\alpha+2}{p} = 1$ and $p \leq 1$, this time we have by (3.8)

$$A_{\alpha, s}^p \subset C^1(\overline{\mathbf{D}}) \quad (3.25)$$

in place of (3.10). We introduce decomposition $J_m = J_m^1 + J_m^2$ and auxiliary exponents σ_k 's, σ , λ adjusted to this setting. This time we let $J_m^1 = (j_1, 0, \dots, 0)$, and as before we may assume $J_m^2 \neq 0$ and we define

$$\frac{1}{\sigma_k} = j_k \left\{ 1 - \frac{p(s-k)}{\alpha+2} \right\} = \frac{j_k(k-1)}{s-1}, \quad k = 2, \dots, m. \quad (3.26)$$

Also, we put

$$\frac{1}{\sigma} = \sum_{k=2}^m \frac{1}{\sigma_k} = \frac{\|J_m\| - |J_m|}{s-1} = \frac{m-1}{s-1}$$

and

$$\frac{1}{\lambda} = 1 - \frac{p(m-1)}{\alpha+2} = 1 - \frac{1}{\sigma}.$$

Note that $1 < \sigma, \lambda < \infty$ and $A_{\alpha, m-1}^p \subset A_{\alpha}^{p\lambda}$. Also, note that $\sigma_k < \infty$ if and only if $j_k \geq 1$ for each $k = 2, \dots, m$. We may thus follow the proof of Sub-Case (a) above and see that the previous estimates (3.19), (3.20) and (3.21) are still available. Meanwhile, we have by (3.25)

$$\left\| (g')^{j_1} \right\|_{\infty} \lesssim \|g\|_{A_{\alpha, s}^p}^{j_1}.$$

This completes the proof of Case (ii) and the proof of the lemma. $\square$

Now we are ready for the proof of Theorem 1.4. We actually prove a bit strengthened version as in the next theorem.

Theorem 3.3. *Suppose $(s-1, p, \alpha) \in \mathcal{R}$ and assume that s is a positive integer if $\alpha = -1$. If $\varphi \in A_{\alpha, s}^p$, then C_{φ} is bounded on $A_{\alpha, s-k}^p$ for all non-negative integers k less than or equal to s . In particular, C_{φ} is bounded on $A_{\alpha, s}^p$ if and only if $\varphi \in A_{\alpha, s}^p$.*

Proof. If $\alpha > -1$, we choose a positive integer $n \geq s$, and note that $A_{\alpha, s}^p \approx A_{\alpha+p(n-s), n}^p$ by (3.3) and $(n-1, p, \alpha+p(n-s)) \in \mathcal{R}$. Thus, without loss of generality, we may assume that s is a positive integer. It suffices to prove that if $\varphi \in A_{\alpha, s}^p$, $0 \leq k \leq s$, and k is an integer, then C_{φ} is bounded on $A_{\alpha, k}^p$.

There is nothing to prove when $k = 0$, since every composition operator is bounded on A_α^p . Let $k \geq 1$. From the norm equivalence (3.1) and the expansion formula given by Lemma 3.1, it suffices to show that

$$\|f^{(k-m+1)}(\varphi)\varphi^{J_m}\|_{A_\alpha^p} \lesssim \|f\|_{A_{\alpha,k}^p} \|\varphi\|_{A_{\alpha,s}^p}^{k-m+1}, \quad m = 1, \dots, k \quad (3.27)$$

for $f \in A_{\alpha,k}^p$ and J_m with $|J_m| = k - m + 1$ and $\|J_m\| = k$. So, consider such f and J_m .

First consider the case that $m \leq s - 1$, in addition to $1 \leq m \leq k$. Then from Lemma 3.2 and (3.1) we get that

$$\begin{aligned} \|f^{(k-m+1)}(\varphi)\varphi^{J_m}\|_{A_\alpha^p} &\lesssim \|f^{(k-m+1)}\|_{A_{\alpha,m-1}^p} \|\varphi\|_{A_{\alpha,s}^p}^{k-m+1} \\ &\lesssim \|f\|_{A_{\alpha,k}^p} \|\varphi\|_{A_{\alpha,s}^p}^{k-m+1}. \end{aligned}$$

so that (3.27) holds when $m \leq s - 1$.

It remains to consider the possibility that $m = s$. Since also $1 \leq m \leq k \leq s$, we have $m = k = s$ in this case. Note that $A_{\alpha,s-1}^p \subset \Lambda_{s-1-(\alpha+2)/p}$ by (3.6) in case $sp > \alpha + 2 + p$ and $A_{\alpha,s-1}^p \subset H_1^1$ by (3.8) otherwise. In either case we have $f' \in A_{\alpha,s-1}^p \subset C(\overline{D})$ with continuous embedding. Also, note that $\varphi^{J_s} = \varphi^{(s)}$. Thus

$$\|f'(\varphi)\varphi^{(s)}\|_{A_\alpha^p} \lesssim \|f'\|_\infty \|\varphi^{(s)}\|_{A_\alpha^p} \lesssim \|f\|_{A_{\alpha,s}^p} \|\varphi\|_{A_{\alpha,s}^p},$$

which is exactly (3.27) when $m = s$. This completes the proof that C_φ is bounded on $A_{\alpha,k}^p$ and the proof of the theorem. $\square$

Corollary 3.4. *Suppose $(s-2, p, \alpha) \in \mathcal{R}$ and assume that s is a positive integer if $\alpha = -1$. If C_φ is bounded on $A_{\alpha,s}^p$, then C_φ is bounded on $A_{\alpha,t}^p$ for all $t \leq s$.*

Proof. First consider t such that $s-1 \leq t < s$. If C_φ is bounded on $A_{\alpha,s}^p$, then $C_\varphi z = \varphi \in A_{\alpha,s}^p \subset A_{\alpha,t}^p$. Note that $(t-1, p, \alpha) \in \mathcal{R}$, because $t \geq s-1$. So, an application of Theorem 3.3 yields that C_φ is bounded on $A_{\alpha,t}^p$ for this range of t . Since any $t < s$ can be written $t = r-k$, where $s-1 \leq r < s$ and k is a non-negative integer, the result for the full range of t now follows from Theorem 3.3. $\square$

As a by-product, we have the following result showing that $A_{\alpha,s}^p$ is an algebra if $(s, p, \alpha) \in \mathcal{R}$. The case $p = 2$ has been long known from the work of Taylor [13]. It is interesting to see that this can be viewed as the Sobolev analogue of (2.16).

Theorem 3.5. *Suppose $(s, p, \alpha) \in \mathcal{R}$ and assume that s is a positive integer if $\alpha = -1$. Then $\mathcal{M}(A_{\alpha,s}^p) = A_{\alpha,s}^p$.*

Proof. Since $A_{\alpha,s}^p$ contains constant functions, $\mathcal{M}(A_{\alpha,s}^p) \subset A_{\alpha,s}^p$ is clear. Now, we prove $A_{\alpha,s}^p \subset \mathcal{M}(A_{\alpha,s}^p)$. So, let $f, g \in A_{\alpha,s}^p$. As in the proof of Theorem 3.3, we may assume that s is a positive integer. Note that $A_{\alpha,s}^p \subset C(\overline{D})$ by (3.6) and (3.8), because $(s, p, \alpha) \in \mathcal{R}$. So, it suffices to show

$$\|f^{(s-m)}g^{(m)}\|_{A_\alpha^p} \lesssim \|f\|_{A_{\alpha,s}^p} \|g\|_{A_{\alpha,s}^p} \quad (3.28)$$

for $m = 1, \dots, s-1$.

Let G be a primitive of g . Then, for $2 \leq m \leq s$, it follows from (3.1) and Lemma 3.2 that

$$\begin{aligned} \|f^{(s-m+1)}g^{(m-1)}\|_{A_\alpha^p} &= \|f^{(s-m+1)}G^{(m)}\|_{A_\alpha^p} \\ &\lesssim \|f^{(s-m+1)}\|_{A_{\alpha, m-1}^p} \|G\|_{A_{\alpha, s+1}^p} \\ &\lesssim \|f\|_{A_{\alpha, s}^p} \|g\|_{A_{\alpha, s}^p}, \end{aligned}$$

which is (3.28). The proof is complete. $\square$

We now turn to the proof of Theorem 1.6. To this end we need an analogue of Lemma 3.2. Note that the restriction $p \leq 1$ for $sp = \alpha + 2$ is removed in the hypothesis of the next lemma.

Lemma 3.6. *Suppose $sp \geq \alpha + 2$ and assume that s is a positive integer. Let $m \leq s$ be a positive integer. Then there is a constant $C = C(s, p, \alpha, m)$ such that*

$$\|g^{J_m}\|_{A_\alpha^p} \leq C \|g\|_{A_{\alpha, s}^p}^{|J_m|}$$

for functions $g \in A_{\alpha, s}^p$ and multi-indices J_m with $|J_m| + m - 1 = \|J_m\| \leq s$.

Proof. Fix $g \in A_{\alpha, s}^p$ and a multi-index $J_m \neq 0$ with $|J_m| + m - 1 = \|J_m\| \leq s$. Note that $|J_s| = 1$ if $m = s$, because $\|J_s\| = s$, and thus $g^{J_s} = g^{(s)}$. Thus the case $m = s$ is easily treated by (3.1). So, assume $m \leq s - 1$ for the rest of the proof. Also, note that the case $sp > \alpha + 2 + p$ is covered by Lemma 3.2 with $f = 1$. It remains to consider $\alpha + 2 \leq sp \leq \alpha + 2 + p$. Even in such a case the proof is parallel to and simpler than that of Lemma 3.2. Details are included for completeness.

First, consider the case where

$$0 < s - \frac{\alpha + 2}{p} \leq 1. \quad (3.29)$$

We use the same auxiliary exponents σ_k (but with $k \geq 1$) defined by

$$\frac{1}{\sigma_k} = j_k \left\{ 1 - \frac{p(s-k)}{\alpha+2} \right\}, \quad k = 1, \dots, m$$

and the corresponding σ is given by

$$\frac{1}{\sigma} = \sum_{k=1}^m \frac{1}{\sigma_k}.$$

Then, since $m \leq s - 1$ and $a_0 := 1 - \frac{ps}{\alpha+2} < 0$, we have by (3.17) that

$$s \left(1 - \frac{1}{\sigma} \right) \geq |a_0| \sum_{k=1}^m j_k (s-k) \geq |a_0| \|J_m\| > 0.$$

Thus, $1 < \sigma \leq \infty$. Note we have by (3.29) that

$$\sigma_k = \infty \iff j_k = 0 \quad \text{or} \quad k = 1 = s - \frac{\alpha + 2}{p}. \quad (3.30)$$

In the case where $\sigma_1 < \infty$ or $j_1 = 0$, we have the desired estimate by the same argument using Hölder's inequality as in the proof of Sub-Case (a) of Lemma 3.2. In the case where $\sigma_1 = \infty$ and $j_1 \geq 1$, we apply Hölder's inequality as before to get that

$$\|g^{J_m}\|_{A_\alpha^p} \leq \|(g')^{j_1}\|_{A_{\alpha}^{p\sigma'}} \|g^{J'_m}\|_{A_\alpha^{p\sigma}}$$

where $J'_m = (j_2, \dots, j_m)$ and σ' is the conjugate index of σ . On one hand, we have

$$\|g^{J'_m}\|_{A_{\alpha}^{p\sigma}} \lesssim \|g\|_{A_{\alpha,s}^p}^{|J'_m|}$$

as in Sub-Case (b) of the proof of Lemma 3.2 (vacuous if $\sigma = \infty$). On the other hand, since $s - \frac{\alpha+2}{p} = 1$ by (3.30), we have

$$g' \in A_{\alpha,s-1}^p \subset \text{VMOA}$$

by (3.1) and (3.7). So, using continuity of the embedding $A_{\alpha,s-1}^p \subset \text{VMOA} \subset A_{\alpha}^{p\sigma'j_1}$, we have by (3.1) that

$$\|(g')^{j_1}\|_{A_{\alpha}^{p\sigma'}} = \|g'\|_{A_{\alpha}^{p\sigma'j_1}}^{j_1} \lesssim \|g'\|_{A_{\alpha,s-1}^p}^{j_1} \lesssim \|g\|_{A_{\alpha,s}^p}^{j_1}.$$

These two estimates combine to finish the proof of this case.

Next, consider the case $sp = \alpha + 2$. We use the same σ_k 's and σ introduced above. However, since $s = \frac{\alpha+2}{p}$, they reduce to

$$\frac{1}{\sigma_k} = \frac{kj_k}{s} \quad (k = 1, \dots, m) \quad \text{and} \quad \sigma = \frac{s}{\|J_m\|} \geq 1.$$

Now, since $\sigma_k = \infty$ if and only if $j_k = 0$ for each k , we conclude

$$\|g^{J_m}\|_{A_{\alpha}^p} \lesssim \|g^{J_m}\|_{A_{\alpha}^{p\sigma}} \leq \prod_{j_k \geq 1} \left\| [g^{(k)}]^{j_k} \right\|_{A_{\alpha}^{p\sigma_k}} \lesssim \prod_{j_k \geq 1} \|g\|_{A_{\alpha,k}^{p\sigma_k j_k}}^{j_k} \lesssim \|g\|_{A_{\alpha,s}^p}^{|J_m|}$$

by Hölder's inequality, (3.1) and (3.13). The proof is complete. $\square$

We now prove a bit strengthened version of Theorem 1.6 as in the next theorem.

Theorem 3.7. *Let $sp \geq \alpha + 2$ and assume that s is a positive integer if $\alpha = -1$. If $\varphi \in A_{\alpha,s}^p$ and $\|\varphi\|_{\infty} < 1$, then C_{φ} is compact on $A_{\alpha,s-k}^p$ for all non-negative integers k less than or equal to s . In particular, for parameters with $(s, p, \alpha) \in \mathcal{R}$, C_{φ} is compact on $A_{\alpha,s}^p$ if and only if $\varphi \in A_{\alpha,s}^p$ and $\|\varphi\|_{\infty} < 1$.*

Proof. As in the proof of Theorem 3.3, we may assume that s is a positive integer. The proof of sufficiency employs standard arguments, with Lemma 3.6 used for a key estimate. Let $\varphi \in A_{\alpha,s}^p$ be such that $\|\varphi\|_{\infty} < 1$. Let $k \leq s$ be a non-negative integer. We need to prove that C_{φ} is compact on $A_{\alpha,k}^p$. Every composition operator C_{φ} with $\|\varphi\|_{\infty} < 1$ is compact on A_{α}^p which covers the case $k = 0$. So, let $k \geq 1$.

Suppose $\{f_n\}$ is a bounded sequence in $A_{\alpha,k}^p$. To show C_{φ} is compact on $A_{\alpha,k}^p$, we must show that $\{C_{\varphi}f_n\}$ has a convergent subsequence in $A_{\alpha,k}^p$. Since $\{f_n\}$ is a normal family (see [2]), it has a subsequence (which for simplicity we continue to denote $\{f_n\}$) that converges uniformly on compact subsets of $\mathbf{D}$ to some $f \in A_{\alpha,k}^p$. Thus, by subtracting f from each term in the sequence, we may assume that $\{f_n\}$ converges uniformly to 0 on compact subsets of $\mathbf{D}$, and we must show that $\{C_{\varphi}f_n\}$ is norm convergent to zero in $A_{\alpha,k}^p$. By (3.1) and Lemma 3.1, it suffices to show that

$$\|f_n^{(k-m+1)}(\varphi)\varphi^{J_m}\|_{A_{\alpha}^p} \rightarrow 0, \quad m = 1, \dots, k \quad (3.31)$$

for J_m with $|J_m| = k - m + 1$ and $\|J_m\| = k$. But $\|f_n^{(k-m+1)}(\varphi)\|_{\infty} \rightarrow 0$, from the uniform convergence to 0 of $\{f_n\}$ on $\overline{\varphi(\mathbf{D})} \subset \mathbf{D}$, and $\|\varphi^{J_m}\|_{A_{\alpha}^p} \lesssim \|\varphi\|_{A_{\alpha,s}^p}^{k-m+1}$ from Lemma 3.6. These observations combine to give (3.31).

The proof of necessity with $(s, p, \alpha) \in \mathcal{R}$ is easy: If C_φ is compact on $A_{\alpha, s}^p$, then from Theorem 1.1 we get that $\|\varphi\|_\infty < 1$. Also, since a compact operator is bounded, $C_\varphi z = \varphi \in A_{\alpha, s}^p$, which completes the proof of the necessity and completes the proof of the theorem. $\square$

By an argument similar to the proof of Corollary 3.4, we have the following.

Corollary 3.8. *Suppose $(s-1)p \geq \alpha+2$ and assume that s is a positive integer if $\alpha = -1$. If C_φ is compact on $A_{\alpha, s}^p$, then C_φ is compact on $A_{\alpha, t}^p$ for all $t \leq s$.*

We now close this section with a question. It has been noticed that the boundedness of a composition operator on a smaller space often implies the same on larger spaces. The embedding property (3.6) is a good example. It is known that the boundedness of C_φ on $A_{\alpha, s}^p$ with $\alpha+2 < sp < \alpha+2+p$ implies the same on $\Lambda_{s-(\alpha+2)/p}$; see Theorem 3.2 of [2]. Note that the same property (trivially) continues to hold for $sp > \alpha+2+p$ by Theorem 1.3 and Theorem 1.4. So, the only parameters missing from this discussion are those with $sp = \alpha+2+p$. Thus we are led to the next problem.

Problem 3.9. *Let $sp = \alpha+2+p$ and suppose that C_φ is bounded on $A_{\alpha, s}^p$. Does it follow that C_φ is bounded on Λ_1 ? We do not know even for the special cases $A_{p-2, 2}^p$ and $H_{3/2}^2$.*

Note that the compact analogue holds by Theorem 1.5 and Theorem 1.6. Also, note that we have $A_{\alpha, s}^p \subset \text{Lip}_1$ for $p \leq 1$ by (3.8). In that case, using Theorem 3.3 we see that the boundedness of C_φ on $A_{\alpha, s}^p$ implies the same on Lip_1 . A related question is raised by the relation $\text{Lip}_1 \subset \Lambda_1$: *If φ' is bounded, then does (2.9) hold?*

4. EXAMPLES

Recall from the Introduction that when $sp < \alpha+1$ every composition operator is bounded on $A_{\alpha, s}^p$, and from Theorem 1.4 that when $\alpha+2+p < sp$, C_φ is bounded on $A_{\alpha, s}^p$ if and only if $\varphi \in A_{\alpha, s}^p$. In this section we give several examples that illustrate what can happen in the intermediate range $\alpha+1 \leq sp \leq \alpha+2+p$.

4.1. The case $sp = \alpha+2+p$. From Theorem 1.4, we know that if $sp = \alpha+2+p$ and $p \leq 1$, then C_φ is bounded on $A_{\alpha, s}^p$ if and only if $\varphi \in A_{\alpha, s}^p$. Here we present an example that shows this result does not extend to any $p > 1$. We use the following elementary lemma, which can be proved by changing to polar coordinates centered at $z = 1$.

Lemma 4.1. *Let $\alpha > -1$.*

$$\int_{\mathbf{D}} \frac{|\log(1-z)|^\beta}{|1-z|^{\alpha+2}} dA_\alpha(z) < \infty$$

if and only if $\beta < -1$.

Example 4.2. *Let $sp = \alpha+2+p$ with $p > 1$, and let s be an integer if $\alpha = -1$. Then there exists $\varphi \in A_{\alpha, s}^p$ such that C_φ is not bounded on $A_{\alpha, s}^p$.*

Proof. We assume $\alpha > -1$; for $\alpha = -1$ proof is similar. Since $A_{\alpha, s}^p = A_{\alpha+(k-s)p, k}^p$ by (3.3) for some positive integer k , we may assume that s is an integer. Note also that $s \geq 2$, since $s = 1 + (\alpha+2)/p > 1$. For $t \in (0, 1)$, define g_t by

$$g_t(z) = (1-z)[1 - \log(1-z)]^t,$$

and let $\varphi = \psi(g_t)$ where $\psi(z) = (1-z)/(1+z)$. It is easy to check that $g_t : \mathbf{D} \rightarrow \{z : \Re z > 0\}$, and so $\varphi : \mathbf{D} \rightarrow \mathbf{D}$. Note that we have

$$|\varphi'(z)| \approx |g_t'(z)| \approx |\log(1-z)|^t \quad (4.1)$$

and

$$|g_t^{(n)}(z)| \approx \frac{|\log(1-z)|^{t-1}}{|1-z|^{n-1}}, \quad n \geq 2. \quad (4.2)$$

Here, and in the rest of the proof, all the estimates are for $z \in \mathbf{D}$ sufficiently close to 1. Thus we see from Lemma 3.1 that the major term in the expansion of $\varphi^{(n)} = [\psi(g_t)]^{(n)}$ is the term $g_t^{(n)}$ and thus

$$|\varphi^{(n)}(z)| \approx \frac{|\log(1-z)|^{t-1}}{|1-z|^{n-1}}, \quad n \geq 2. \quad (4.3)$$

And in particular, from (3.1) and using that $p(s-1) = \alpha + 2$ and $s \geq 2$, we obtain that

$$\|\varphi\|_{A_{\alpha,s}^p}^p \approx \|g_t\|_{A_{\alpha,s}^p}^p \approx \int_{\mathbf{D}} |g_t^{(s)}|^p dA_{\alpha} \approx \int_{\mathbf{D}} \frac{|\log(1-z)|^{p(t-1)}}{|1-z|^{\alpha+2}} dA_{\alpha}(z).$$

From Lemma 4.1 we now see that $\varphi, g_t \in A_{\alpha,s}^p$ if and only if $p(t-1) < -1$, or equivalently $t < 1 - 1/p$. We now fix t such that

$$0 < t < 1 - \frac{1}{p} < 2t,$$

so $\varphi, g_t \in A_{\alpha,s}^p$.

We now prove $C_{\varphi} g_t \notin A_{\alpha,s}^p$. First, note that

$$|\log(1-\varphi(z))| \approx |\log g_t(z)| \approx |\log(1-z)|. \quad (4.4)$$

Also, we have $|1-\varphi(z)| \approx |g_t(z)|$. So, for $1 \leq m \leq s-1$, we have by (4.2)

$$\begin{aligned} |g_t^{(s-m+1)}(\varphi(z))| &\approx \frac{|\log(1-\varphi(z))|^{t-1}}{|1-\varphi(z)|^{s-m}} \\ &\approx \frac{|\log g_t(z)|^{t-1}}{|g_t(z)|^{s-m}} \\ &\approx \frac{|\log(1-z)|^{t-1}}{[|1-z| \cdot |\log(1-z)|^t]^{s-m}} \\ &\approx \frac{|\log(1-z)|^{2t-1-t(s-m+1)}}{|1-z|^{s-m}}. \end{aligned}$$

Also, for a multi-index $J_m = (j_1, \dots, j_m)$ as in Lemma 3.1, we have

$$\begin{aligned} |\varphi^{J_m}(z)| &\approx |\log(1-z)|^{j_1+(t-1)j_1} \cdot \frac{|\log(1-z)|^{(t-1)j_2}}{|1-z|^{(2-1)j_2}} \cdots \frac{|\log(1-z)|^{(t-1)j_m}}{|1-z|^{(m-1)j_m}} \\ &\approx \frac{|\log(1-z)|^{j_1+(t-1)(s-m+1)}}{|1-z|^{m-1}}. \end{aligned}$$

Combining these two estimates, we get

$$\begin{aligned} |g_t^{(s-m+1)}(\varphi(z))| \cdot |\varphi^{J_m}(z)| &\approx \frac{|\log(1-z)|^{(2t-1)+(j_1-|J_m|)}}{|1-z|^{s-1}} \\ &= \frac{|\log(1-z)|^{2t-1}}{|1-z|^{s-1}} (1+o(1)) \end{aligned}$$

for $1 \leq m \leq s-1$. Meanwhile, for $m = s$, we see from (4.1), (4.3) and (4.4) that

$$|g'_t(\varphi(z))| \cdot |\varphi^{(s)}(z)| \approx |\log(1-\varphi(z))|^t \cdot \frac{|\log(1-z)|^{t-1}}{|1-z|^{s-1}} \approx \frac{|\log(1-z)|^{2t-1}}{|1-z|^{s-1}}.$$

This shows that the major term in the expansion of $[g_t(\varphi)]^{(s)}$ behaves like the last quantity above and thus

$$\int_{\mathbf{D}} |(g_t \circ \varphi)^{(s)}|^p dA_\alpha \approx \int_{\mathbf{D}} \frac{|\log(1-z)|^{p(2t-1)}}{|1-z|^{\alpha+2}} dA_\alpha(z).$$

Since $2t > 1 - 1/p$, we have $p(2t-1) > -1$, and so this integral diverges by Lemma 4.1. Thus $C_{\varphi g_t} \notin A_{\alpha,s}^p$ by (3.1), and the proof is complete. $\square$

4.2. The case $\alpha + 2 < sp < \alpha + 2 + p$. Here we show that Theorem 1.4 does not extend to this range. We use another elementary lemma which can be seen by changing to polar coordinates centered at $z = 1$; see Theorem 1.7 of [7].

Lemma 4.3. *Let $\epsilon > 0$. Then $(1-z)^{-\epsilon} \in A_\alpha^p$ if and only if $\epsilon p < \alpha + 2$.*

Example 4.4. *Let $\alpha + 2 < sp < \alpha + 2 + p$ and let s be an integer if $\alpha = -1$. Then there exists $\varphi \in A_{\alpha,s}^p$ such that C_φ is not bounded on $A_{\alpha,s}^p$.*

Proof. We assume $\alpha > -1$; for $\alpha = -1$ proof is similar. Since $A_{\alpha,s}^p = A_{\alpha+(k-s)p,k}^p$ for some positive integer $k \geq s$, we may assume that s is an integer. The assumption $\alpha + 2 < sp < \alpha + 2 + p$ allows us to choose $t \in (0, 1)$ such that

$$\alpha + 2 + t^2 p = sp < \alpha + 2 + tp. \quad (4.5)$$

Fix such a t and let

$$f(z) = (1-z)^t \quad \text{and} \quad \varphi(z) = \frac{1-f(z)}{1+f(z)}.$$

Then it is easily seen that $\varphi : \mathbf{D} \rightarrow \mathbf{D}$ and

$$C_\varphi f(z) = \psi(f(z))(1-z)^{t^2} \quad \text{where} \quad \psi(z) = \left(\frac{2}{1+z} \right)^t.$$

Clearly, we have $|f^{(s)}(z)| \approx |1-z|^{t-s}$ and thus $f \in A_{\alpha,s}^p$ by (3.1), (4.5) and Lemma 4.3. A straightforward calculation using Lemma 3.1 gives

$$\begin{aligned} (C_\varphi f)^{(s)}(z) &= c\psi(f(z))(1-z)^{t^2-s} \\ &\quad + \sum_{j=1}^s \sum_{m=1}^j c_{jm} \psi^{(j-m+1)}(f(z))(1-z)^{t(j-m+1)+t^2-s} \end{aligned}$$

for some coefficients $c = c(s, t)$ and $c_{jm} = c_{jm}(s, t)$. Since functions $\psi^{(j-m+1)}(f)$ have no singularity, it is easily seen from the above that

$$|(C_\varphi f)^{(s)}(z)| \approx |1-z|^{t^2-s}.$$

A similar and simpler calculation yields

$$|\varphi^{(s)}(z)| \approx |1 - z|^{t-s}.$$

We now see from (3.1), (4.5) and Lemma 4.3 that $\varphi \in A_{\alpha,s}^p$, but $C_\varphi f \notin A_{\alpha,s}^p$. This completes the proof. $\square$

4.3. The case $\alpha + 1 \leq sp \leq \alpha + 2$. It is known that if φ is of bounded valence then C_φ is always bounded on $A_{\alpha,s}^p$ with $\alpha > -1$, $p \geq 2$ and $sp \leq \alpha + 2$. Also, there is a similar but partial result for $p < 2$. See Theorem 1.2 of [2]. The situation here is thus much more delicate, and we restrict our attention to parameters with $p = 2$, $\alpha > -1$, and $s = 1$, i.e. to the weighted Dirichlet spaces $A_{\alpha,1}^2$. An example relevant to our work can be found in [9], where composition operators on these spaces were studied. It corresponds to the case $\alpha + 1 < sp < \alpha + 3/2$.

Example 4.5. [9, Proposition 3.12] *There exists $\varphi \in \bigcap \{A_{\alpha,1}^2 : \alpha > 1/2\}$ for which C_φ is not bounded on $A_{\alpha,1}^2$ for any $\alpha < 1$.*

We have no examples for the range $\alpha + 3/2 \leq sp < \alpha + 2$. Our final example is for the classical (unweighted) Dirichlet space $A_{0,1}^2$, which corresponds to the case $sp = \alpha + 2$. We will use a Carleson-measure-type characterization from [9] of when C_φ is bounded on a weighted Dirichlet space. For $0 < \delta \leq 2$ and $\zeta \in \partial\mathbf{D}$, let

$$S(\delta, \zeta) = \{z \in \mathbf{D} : |z - \zeta| < \delta\}$$

denote the usual Carleson box. Then, stated here only for $\alpha = 0$, we have the following characterization:

Lemma 4.6. [9, Proposition 5.1] *Let $\varphi \in A_{0,1}^2$. Then C_φ is bounded on $A_{0,1}^2$ if and only if*

$$\sup \left\{ \delta^{-2} \int_{\varphi^{-1}(S(\delta, \zeta))} |\varphi'|^2 dA : 0 < \delta \leq 2, \zeta \in \partial\mathbf{D} \right\} < \infty.$$

Example 4.7. *There exists $\varphi \in A_{0,1}^2$ for which C_φ is not bounded on $A_{0,1}^2$.*

Proof. Let $\psi : \mathbf{D} \rightarrow \mathbf{G}$ be a Riemann map, where

$$\mathbf{G} = \{z : 1 < \Re z < \infty, 0 < \Im z < (\Re z)^{-2}\},$$

and let $\varphi = e^{i\psi}$ so that $\varphi : \mathbf{D} \rightarrow \mathbf{D}$. Then

$$\|\varphi\|_{A_{0,1}^2}^2 = \int_{\mathbf{D}} |\varphi'|^2 dA \leq \int_{\mathbf{D}} |\psi'|^2 dA = \text{Area}(\mathbf{G}) < \infty,$$

and so $\varphi \in A_{0,1}^2$. Since $|\varphi| \approx 1$ on $\mathbf{D}$, we have $|\varphi'| = |\psi'| |\varphi| \approx |\psi'|$. Hence

$$\int_{\varphi^{-1}(S(\delta, 1))} |\varphi'|^2 dA \approx \int_{\varphi^{-1}(S(\delta, 1))} |\psi'|^2 dA = \text{Area}(R(\delta)), \quad (4.6)$$

where $R(\delta) = \{w : e^{iw} \in S(\delta, 1)\}$. When $\delta > 0$ is small, there are approximately $1/(2\pi\sqrt{\delta})$ components of $R(\delta)$ that are mapped by e^{iz} onto $S(\delta, 1)$, and each of these components has area comparable to δ^2 . Hence

$$\text{Area}(R(\delta)) \gtrsim \delta^{3/2}.$$

Combining this with (4.6), we see from Lemma 4.6 that C_φ is not bounded on $A_{0,1}^2$. $\square$

We remark that this example can be easily modified to work for $p > 1$, but still with $s = 1$ and $sp = \alpha + 2$.

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DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, SEOUL 136–701, KOREA
E-mail address: `cbr@korea.ac.kr`

DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, SEOUL 136–701, KOREA
E-mail address: `koohw@korea.ac.kr`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII, HONOLULU, HAWAII 96822
E-mail address: `wayne@math.hawaii.edu`