1 The logarithm function

The natural logarithm function, \( \ln x = \log_e x \), is the inverse of the exponential function \( e^x \). That is, \( \ln(e^x) = x \) and \( e^{\ln x} = x \).

Exercise 1.

a. Use the Maxima Help index to find out how you would call the natural logarithm function in Maxima.
   Write the name of the function here: ________________

b. Maxima does not have a built-in function for the base 10 logarithm or other bases. Suppose you want to plot the function \( \log_{10} x \). Recall that \( x = b^{\log_b x} \) holds for any number \( b \). Taking the natural log of both sides of this equality gives

\[
\ln x = \ln b^{\log_b x} \quad \text{or} \quad \ln x = (\log_b x)(\ln b).
\]

Dividing both sides of the last equation by \( \ln b \), we have

\[
\log_b x = \frac{\ln x}{\ln b}.
\]

How would you define a base 10 logarithm function in Maxima? Name this function \( \log_{10} \), and write your definition here:

\[
\log_{10}(x) := ________________
\]

Define a base 2 logarithm function. Write your definition here:

\[
\log_{2}(x) := ________________
\]

Finally, define these functions in Maxima.

c. Using the functions you defined above, find the decimal value of the following:
   (N.B. to get a decimal answer in Maxima, use the float command.)

\[
\log_{10}(10^8) = __________
\]

\[
\log_{2}(\log_{10}(10^8)) = __________
\]
Exercise 2.

a. Given a function \( f(x) \), you can define a secant line function for \( f(x) \) using the equation:

\[
SL(a, h) = f(a) + \frac{f(a + h) - f(a)}{h}(x - a)
\]

As you can see, this is the line passing through the point \( (a, f(a)) \), and having slope \( \frac{f(a + h) - f(a)}{h} \). If the graph of \( f(x) \) is a smooth curve, then as \( h \) gets smaller, the slope of the line \( SL(a, h) \) will approach \( f'(a) \), which is the slope of the line tangent to the curve \( f(x) \) at the point \( (a, f(a)) \). That is, when \( f \) is smooth (differentiable),

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

Define a secant line function for the function \( \log_2 \) by typing the following into Maxima:

\[
SL(a, h) := \log_2(a) + \frac{\log_2(a+h)-\log_2(a)}{h} \times (x-a)
\]

b. Plot the curve \( \log_2(x) \) and the secant line passing through the point \( (1, \log_2(1)) \) on the same graph, using Maxima’s Plot2D button or using the Maxima command line. (Hint: You want to put \( a = 1 \) in your \( SL(a, h) \) function. For now, just set \( h \) to whatever you like, say \( h=1 \).)

c. To zoom in on the point \( (1, 0) \), you can adjust the ranges of \( x \) and \( y \) values in the \texttt{wxplot2d} command by manually editing the input line containing the \texttt{wxplot2d} command (i.e. you don’t have to click the Plot2d button every time.) Also, you can delete the first two letters of the command, and try entering just \texttt{plot2d} plus the rest of the command line. This will open up a separate gnu plot window that you can make larger and use the cursor to identify the coordinates of various points on the graph.

d. By adjusting the value of \( h \) in the \( SL(a, h) \) function, you can see the secant line approaching the tangent line. Type the following into Maxima:

\[
\texttt{wxplot2d([ SL(1,2), SL(1,1/2), log2(x)], [x,.5,2], [y,-1,1])}
\]

e. Finally, type the following into Maxima:

\[
\texttt{wxplot2d([ SL(1,1/1000), log2(x)], [x,.5,1.6], [y,-1,1])}
\]

You should observe two distinct lines. However, the property of the tangent line is that, when you zoom in, the lines become indistinguishable. Try this:

\[
\texttt{wxplot2d([ SL(1,1/1000), log2(x)], [x,.9,1.1], [y,-1,1])}
\]