8.3.25. Evaluate the integral

\[ \int \frac{8 \, dx}{(4x^2 + 1)^2} \]

**Solution:**

We started this problem correctly in class, but I made a mistake during the computation, so here is the full solution (with many more details than you need to write, once you are comfortable with these problems).

The integral is

\[ \int \frac{8 \, dx}{(4x^2 + 1)^2} = \int \frac{8 \, dx}{(4(x^2 + \frac{1}{4}))^2} = \int \frac{8 \, dx}{16(x^2 + \frac{1}{4})^2} = \frac{1}{2} \int \frac{dx}{(x^2 + \frac{1}{4})^2}. \]

The substitution rule on the top of page 453 tells us how to deal with this kind of integral. We identify \( a^2 = \frac{1}{4}, \) or \( a = \frac{1}{2}, \) so

Let \( x = \frac{1}{2} \tan \theta. \) Then \( dx = \frac{1}{2} \sec^2 \theta \, d\theta, \) and

\[ \left( x^2 + \frac{1}{4} \right)^2 = \left( \left( \frac{1}{2} \tan \theta \right)^2 + \frac{1}{4} \right)^2 = \left( \frac{1}{4} \tan^2 \theta + \frac{1}{4} \right)^2 = \frac{1}{16}(\tan^2 \theta + 1)^2 = \frac{1}{16}(\sec^2 \theta)^2 = \frac{1}{16} \sec^4 \theta. \]

So the integral becomes

\[ \frac{1}{2} \int \frac{dx}{(x^2 + \frac{1}{4})^2} = \frac{1}{2} \int \frac{(1/2) \sec^2 \theta \, d\theta}{(1/16) \sec^4 \theta} = \frac{16}{2} \cdot 2 \int \frac{\sec \theta \, d\theta}{\sec^2 \theta} = 4 \int \frac{d\theta}{\sec \theta} = 4 \int \cos^2 \theta \, d\theta. \]

To evaluate \( \int \cos^2 \theta \, d\theta, \) use the trigonometric identity

\[ \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \]

to write

\[ 4 \int \cos^2 \theta \, d\theta = 4 \int \frac{1 + \cos 2\theta}{2} \, d\theta, \]

which is equal to

\[ 2 \int 1 + \cos 2\theta \, d\theta = 2\theta + \sin 2\theta + C = 2\theta + 2\sin \theta \cos \theta + C. \]

To get the last equality, recall the two formulas I suggested you memorize:

\[ \sin(A + B) = \sin A \cos B + \sin B \cos A \quad \text{and} \quad \cos(A + B) = \cos A \cos B - \sin A \sin B. \]

If we let \( A = B = \theta \) in the first of these, we get the “double angle” formula

\[ \sin 2\theta = 2 \sin \theta \cos \theta. \]

continued on next page...
To finish the solution, we must write $2\theta + 2\sin \theta \cos \theta + C$ in terms of $x$, using our substitution, $x = \frac{1}{2} \tan \theta$. First, to replace $\theta$, we use

$$x = \frac{1}{2} \tan \theta \iff 2x = \tan \theta \iff \tan^{-1}(2x) = \theta.$$ 

So $2\theta = 2\tan^{-1}(2x)$.

Finally, to write $2\sin \theta \cos \theta$ in terms of $x$, draw a triangle. Since tangent is opposite over adjacent, and $\tan \theta = 2x = \frac{2x}{1}$, label the opposite edge $2x$ and label the adjacent edge $1$. Then the hypotenuse must be $\sqrt{4x^2 + 1}$, by the Pythagorean theorem. Since sine is opposite over hypotenuse, $\sin \theta = \frac{2x}{\sqrt{4x^2 + 1}}$. Cosine is adjacent over hypotenuse, so $\cos \theta = \frac{1}{\sqrt{4x^2 + 1}}$. Putting it all together,

$$2\theta + 2\sin \theta \cos \theta + C = 2\tan^{-1}(2x) + 2 \cdot \frac{2x}{\sqrt{4x^2 + 1}} \cdot \frac{1}{\sqrt{4x^2 + 1}} + C = 2\tan^{-1}(2x) + \frac{4x}{4x^2 + 1} + C.$$ 

Therefore, the answer is

$$\int \frac{8 \, dx}{(4x^2 + 1)^2} = 2\tan^{-1}(2x) + \frac{4x}{4x^2 + 1} + C.$$ 

Incidentally, you can check this answer using Maxima. Use the Calculus menu, select Integrate and enter the expression $8/(4*x^2 + 1)^2$ (don’t forget the *!). Maxima says the answer is

$$8\left(\frac{\text{atan}(2x)}{4} + \frac{x}{8x^2 + 2}\right),$$ 

which, except for the constant $C$, is the same as our answer.