SUMMARY

Two useful probability bounds are provided by the Markov and Chebyshev inequalities. The Markov inequality is concerned with nonnegative random variables, and says that for $X$ of that type

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

for every positive value $a$. The Chebyshev inequality, which is a simple consequence of the Markov inequality, states that if $X$ has mean $\mu$ and variance $\sigma^2$, then for every positive $k$,

$$P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}$$

The two most important theoretical results in probability are the central limit theorem and the strong law of large numbers. Both are concerned with a sequence of independent and identically distributed random variables. The central limit theorem says that if the random variables have a finite mean $\mu$ and a finite variance $\sigma^2$, then the distribution of the sum of the first $n$ of them is, for large $n$, approximately that of a normal random variable with mean $n\mu$ and variance $n\sigma^2$. That is, if $X_i$, $i \geq 1$, is the sequence, then the central limit theorem states that for every real number $a$,

$$\lim_{n \to \infty} P\left\{\frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}} \leq a\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} \, dx$$

The strong law of large numbers requires only that the random variables in the sequence have a finite mean $\mu$. It states that with probability 1, the average of the first $n$ of them will converge to $\mu$ as $n$ goes to infinity. This implies that if $A$ is any specified event of an experiment for which independent replications are performed, then the limiting proportion of experiments whose outcomes are in $A$ will, with probability 1, equal $P(A)$. Therefore, if we accept the interpretation that "with probability 1" means "with certainty," we obtain the theoretical justification for the long-run relative frequency interpretation of probabilities.

PROBLEMS

1. Suppose that $X$ is a random variable with mean and variance both equal to 20. What can be said about $P\{0 \leq X \leq 40\}$?

2. From past experience a professor knows that the test score of a student taking her final examination is a random variable with mean 75.
   (a) Give an upper bound for the probability that a student's test score will exceed 85.
   Suppose, in addition, the professor knows that the variance of a student's test score is equal to 25.
(b) What can be said about the probability that a student will score between 65 and 85?
(c) How many students would have to take the examination to ensure, with probability at least .9, that the class average would be within 5 of 75? Do not use the central limit theorem.

3. Use the central limit theorem to solve part (c) of Problem 2.
4. Let \( X_1, \ldots, X_{20} \) be independent Poisson random variables with mean 1.
   (a) Use the Markov inequality to obtain a bound on
   \[
P\left( \sum_{i=1}^{20} X_i > 15 \right)
   \]
   (b) Use the central limit theorem to approximate \( P\left( \sum_{i=1}^{20} X_i > 15 \right) \).
5. Fifty numbers are rounded off to the nearest integer and then summed. If the individual round-off errors are uniformly distributed over \((-0.5, 0.5)\) what is the probability that the resultant sum differs from the exact sum by more than 3?
6. A die is continually rolled until the total sum of all rolls exceeds 300. What is the probability that at least 80 rolls are necessary?
7. One has 100 light bulbs whose lifetimes are independent exponentials with mean 5 hours. If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one, what is the probability that there is still a working bulb after 525 hours?
8. In Problem 7 suppose that it takes a random time, uniformly distributed over \((0, 0.5)\), to replace a failed bulb. What is the probability that all bulbs have failed by time 550?
9. If \( X \) is a gamma random variable with parameters \((n, 1)\) how large need \( n \) be so that
   \[
P\left( \left| \frac{X}{n} - 1 \right| > .01 \right) < .01
   \]
10. Civil engineers believe that \( W \), the amount of weight (in units of 1000 pounds) that a certain span of a bridge can withstand without structural damage resulting, is normally distributed with mean 400 and standard deviation 40. Suppose that the weight (again, in units of 1000 pounds) of a car is a random variable with mean 3 and standard deviation .3. How many cars would have to be on the bridge span for the probability of structural damage to exceed .1?
11. Many people believe that the daily change of price of a company’s stock on the stock market is a random variable with mean 0 and variance \( \sigma^2 \). That is, if \( Y_n \) represents the price of the stock on the \( n \)th day, then
   \[
   Y_n = Y_{n-1} + X_n, \quad n \geq 1
   \]
   where \( X_1, X_2, \ldots \) are independent and identically distributed random variables with mean 0 and variance \( \sigma^2 \). Suppose that the stock’s price today is 100.
If $\sigma^2 = 1$, what can you say about the probability that the stock’s price will exceed 105 after 10 days?

12. We have 100 components that we will put in use in a sequential fashion. That is, component 1 is initially put in use, and upon failure it is replaced by component 2, which is itself replaced upon failure by component 3, and so on. If the lifetime of component $i$ is exponentially distributed with mean $10 + i/10$, $i = 1, \ldots, 100$, estimate the probability that the total life of all components will exceed 1200. Now repeat when the life distribution of component $i$ is uniformly distributed over $(0, 20 + i/5)$, $i = 1, \ldots, 100$.

13. Student scores on exams given by a certain instructor have mean 74 and standard deviation 14. This instructor is about to give two exams, one to a class of size 25 and the other to a class of size 64.
(a) Approximate the probability that the average test score in the class of size 25 exceeds 80.
(b) Repeat part (a) for the class of size 64.
(c) Approximate the probability that the average test score in the larger class exceeds that of the other class by over 2.2 points.
(d) Approximate the probability that the average test score in the smaller class exceeds that of the other class by over 2.2 points.

14. A certain component is critical to the operation of an electrical system and must be replaced immediately upon failure. If the mean lifetime of this type of component is 100 hours and its standard deviation is 30 hours, how many of these components must be in stock so that the probability that the system is in continual operation for the next 2000 hours is at least .95?

15. An insurance company has 10,000 automobile policyholders. The expected yearly claim per policyholder is $240 with a standard deviation of $800. Approximate the probability that the total yearly claim exceeds $2.7 million.

16. Redo Example 5b under the assumption that the number of man–woman pairs is (approximately) normally distributed. Does this seem like a reasonable supposition?

17. Repeat part (a) of Problem 2 when it is known that the variance of a student’s test score is equal to 25.

18. A lake contains 4 distinct types of fish. Suppose that each fish caught is equally likely to be any one of these types. Let $Y$ denote the number of fish that need be caught to obtain at least one of each type.
(a) Give an interval $(a, b)$ such that $P\{a \leq Y \leq b\} \approx .90$.
(b) Using the one-sided Chebyshev inequality, how many fish need we plan on catching so as to be at least 90 percent certain of obtaining at least one of each type?

19. If $X$ is a nonnegative random variable with mean 25, what can be said about:
(a) $E[X^3]$;
(b) $E[\sqrt{X}]$;
(c) $E[\log X]$;
(d) $E[e^{-X}]$?
20. Let $X$ be a nonnegative random variable. Prove that

$$E[X] \leq (E[X^2])^{1/2} \leq (E[X^3])^{1/3} \leq \cdots$$

21. Would the results of Example 5f change if the investor were allowed to divide her money and invest the fraction $\alpha$, $0 < \alpha < 1$ in the risky proposition and invest the remainder in the risk-free venture? Her return for such a split investment would be $R = \alpha X + (1 - \alpha)m$.

22. Let $X$ be a Poisson random variable with mean 20.
   (a) Use the Markov inequality to obtain an upper bound on
   $$p = P\{X \geq 26\}$$
   (b) Use the one-sided Chebyshev inequality to obtain an upper bound on $p$.
   (c) Use the Chernoff bound to obtain an upper bound on $p$.
   (d) Approximate $p$ by making use of the central limit theorem.
   (e) Determine $p$ by running an appropriate program.

**THEORETICAL EXERCISES**

\(\star\) 1. If $X$ has variance $\sigma^2$, then $\sigma$, the positive square root of the variance, is called the *standard deviation*. If $X$ has mean $\mu$ and standard deviation $\sigma$, show that

$$P\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

2. If $X$ has mean $\mu$ and standard deviation $\sigma$, the ratio $r = |\mu|/\sigma$ is called the *measurement signal-to-noise ratio* of $X$. The idea is that $X$ can be expressed as $X = \mu + (X - \mu)$ with $\mu$ representing the signal and $X - \mu$ the noise. If we define $|(X - \mu)/\mu| \equiv D$ as the relative deviation of $X$ from its signal (or mean) $\mu$, show that for $\alpha > 0$,

$$P\{D \leq \alpha\} \geq 1 - \frac{1}{r^2\alpha^2}$$

3. Compute the measurement signal-to-noise ratio—that is, $|\mu|/\sigma$ where $\mu = E[X]$, $\sigma^2 = \text{Var}(X)$—of the following random variables:
   (a) Poisson with mean $\lambda$;
   (b) binomial with parameters $n$ and $p$;
   (c) geometric with mean $1/p$;
   (d) uniform over $(a, b)$;
   (e) exponential with mean $1/\lambda$;
   (f) normal with parameters $\mu$, $\sigma^2$.

4. Let $Z_n, n \geq 1$ be a sequence of random variables and $c$ a constant such that for each $\varepsilon > 0$, $P\{|Z_n - c| > \varepsilon\} \to 0$ as $n \to \infty$. Show that for any bounded continuous function $g$,

$$E[g(Z_n)] \to g(c) \quad \text{as} \quad n \to \infty$$