Chapter 8

Problems

1. \[ P\{0 \leq X \leq 40\} = 1 - P\{|X - 20| > 20\} \geq 1 - 20/400 = 19/20 \]

2. (a) \[ P\{X \geq 85\} \leq E[X]/85 = 15/17 \]
   (b) \[ P\{65 \leq X \leq 85\} = 1 - P\{|X - 75| > 10\} \geq 1 - 25/100 \]
   (c) \[ P\left\{\sum_{i=1}^{n} X_i/n - 75 > 5\right\} \leq \frac{25}{25n} \] so need \( n = 10 \)

3. Let \( Z \) be a standard normal random variable. Then,
   \[ P\left\{\sum_{i=1}^{n} X_i/n - 75 > 5\right\} \approx P\{|Z| > \sqrt{n}\} \leq .1 \] when \( n = 3 \)

4. (a) \[ P\left\{\sum_{i=1}^{20} X_i > 15\right\} \leq 20/15 \]
   (b) \[ P\left\{\sum_{i=1}^{20} X_i > 15\right\} = P\left\{\sum_{i=1}^{20} X_i > 15.5\right\} \]
   \[ \approx P\left\{Z > \frac{15.5 - 20}{\sqrt{20}}\right\} \]
   \[ = P\{Z > -1.006\} \approx .8428 \]

5. Letting \( X_i \) denote the \( i \)th roundoff error it follows that \( E\left[\sum_{i=1}^{50} X_i\right] = 0, \)
   \[ \text{Var}\left[\sum_{i=1}^{50} X_i\right] = 50 \text{Var}(X_i) = 50/12, \]
   where the last equality uses that \( .5 + X \) is uniform \((0, 1)\) and so \( \text{Var}(X) = \text{Var}(.5 + X) = 1/12. \) Hence,
   \[ P\left\{\sum_{i=1}^{50} X_i > 3\right\} \approx P\{ |N(0, 1)| > 3(12/50)^{1/2}\} \]
   by the central limit theorem
   \[ = 2P\{N(0, 1) > 1.47\} = .1416 \]

6. If \( X_i \) is the outcome of the \( i \)th roll then \( E[X] = 7/2 \) \( \text{Var}(X_i) = 35/12 \) and so
   \[ P\left\{\sum_{i=1}^{79} X_i \leq 300\right\} = P\left\{\sum_{i=1}^{79} X_i \leq 300.5\right\} \]
   \[ \approx P\{N(0,1) \leq \frac{300.5 - 79(7/2)}{(79 \times 35/12)^{1/2}}\}
   = P\{N(0,1) \leq 1.58\} = .9429 \]
7. \[ P\left\{ \sum_{i=1}^{100} X_i > 525 \right\} \approx P\left\{ N(0,1) > \frac{525 - 500}{\sqrt{100 \times 25}} \right\} = P\{N(0,1) > .5\} = .3085 \]

where the above uses that an exponential with mean 5 has variance 25.

8. If we let \( X_i \) denote the life of bulb \( i \) and let \( R_i \) be the time to replace bulb \( i \) then the desired probability is \[ P\left\{ \sum_{i=1}^{100} X_i +\sum_{i=1}^{100} R_i \leq 550 \right\} \]. Since \( \sum_{i=1}^{100} X_i +\sum_{i=1}^{100} R_i \) has mean \( 100 \times 5 + 99 \times .25 = 524.75 \) and variance \( 2500 + 99/48 = 2502 \) it follows that the desired probability is approximately equal to \[ P\{N(0,1) \leq [550 - 524.75]/(2502)^{1/2}\} = P\{N(0,1) \leq .505\} = .693 \]

It should be noted that the above used that \( \text{Var}(R_i) = \text{Var}\left(\frac{1}{2}\text{Unif}[0,1]\right) = 1/48 \)

9. Use the fact that a gamma \((n, 1)\) random variable is the sum of \( n \) independent exponentials with rate 1 and thus has mean and variance equal to \( n \), to obtain:

\[
P\left\{ \frac{X-n}{n} > .01 \right\} = P\left\{ \frac{|X-n|}{\sqrt{n}} > .01\sqrt{n} \right\}
\]
\[
\approx P\left\{ |N(0,1)| > .01\sqrt{n} \right\}
\]
\[
= 2P\{N(0,1) > .01\sqrt{n}\}
\]

Now \( P\{N(0,1) > 2.58\} = .005 \) and so \( n = (258)^2 \).

10. If \( W_n \) is the total weight of \( n \) cars and \( A \) is the amount of weight that the bridge can withstand then \( W_n - A \) is normal with mean \( 3n - 400 \) and variance \( .09n + 1600 \). Hence, the probability of structural damage is

\[
P\{W_n - A \geq 0\} \approx P\left\{ Z \geq (400 - 3n)/\sqrt{.09n + 1600} \right\}
\]

Since \( P\{Z \geq 1.28\} = .1 \) the probability of damage will exceed .1 when \( n \) is such that

\[
400 - 3n \leq 1.28 \sqrt{.09n + 1600}
\]

The above will be satisfied whenever \( n \geq 117 \).

12. Let \( L_i \) denote the life of component \( i \).

\[
E\left[ \sum_{i=1}^{100} L_i \right] = 1000 + \frac{1}{10}50(101) = 1505
\]

\[
\text{Var}\left( \sum_{i=1}^{100} L_i \right) = \sum_{i=1}^{100} \left( 10 + \frac{i}{10} \right)^2 = (100)^2 + (100)(101) + \frac{1}{10}\sum_{i=1}^{100} i^2
\]

Now apply the central limit theorem to approximate.
13. (a) \( P\{X > 80\} = P\left( \frac{X - 74}{14/5} > 15/7 \right) \approx P\{Z > 2.14\} \approx .0162 \)

(b) \( P\{Y > 80\} = P\left( \frac{Y - 74}{14/8} > 24/7 \right) \approx P\{Z > 3.43\} = .0003 \)

(c) Using that \( SD(\bar{Y} - \bar{X}) = \sqrt{196/64 + 196/25} \approx 3.30 \) we have
\[
P\{\bar{Y} - \bar{X} > 2.2\} = P\{\bar{Y} - \bar{X}/3.30 > 2.2/3.30\} \approx P\{Z > .67\} = .2514
\]

(d) same as in (c)

14. Suppose \( n \) components are in stock. The probability they will last for at least 2000 hours is
\[
p = P\left( \sum_{i=1}^{n} X_i \geq 2000 \right) \approx P\left( Z \geq \frac{2000 - 100n}{30\sqrt{n}} \right)
\]
where \( Z \) is a standard normal random variable. Since \( .95 = P\{Z \geq -1.64\} \) it follows that \( p \geq .95 \) if
\[
\frac{2000 - 100n}{30\sqrt{n}} \leq -1.64
\]
or, equivalently,
\[
(2000 - 100n)/\sqrt{n} \leq -49.2
\]
and this will be the case if \( n \geq 23 \).

15. \( P\left( \sum_{i=1}^{10,000} X_i > 2,700,000 \right) \approx P\{Z > (2,700,000 - 2,400,000)/(800 \cdot 100)\} = P\{Z \geq 3.75\} \approx 0 \)

18. Let \( Y_i \) denote the additional number of fish that need to be caught to obtain a new type when there are at present \( i \) distinct types. Then \( Y_i \) is geometric with parameter \( \frac{4 - i}{4} \).
\[
E[Y] = E\left[ \sum_{i=0}^{3} Y_i \right] = 1 + \frac{4}{3} + \frac{4}{2} + 4 = \frac{25}{3}
\]
\[
\text{Var}[Y] = \text{Var}\left( \sum_{i=0}^{3} Y_i \right) = \frac{4}{9} + 2 + 12 = \frac{130}{9}
\]
Hence,

\[
P \left[ \left| Y - \frac{25}{3} \right| \geq \frac{25}{3} \sqrt{\frac{1300}{9}} \right] \leq \frac{1}{10}
\]

and so we can take \( a = \frac{25 - \sqrt{1300}}{3}, \ b = \frac{25 + \sqrt{1300}}{3} \).

Also,

\[
P \left[ Y - \frac{25}{3} > a \right] \leq \frac{130}{130 + 9a^2} = \frac{1}{10} \text{ when } a = \frac{\sqrt{1170}}{3}.
\]

Hence \( P \left[ Y > \frac{25 + \sqrt{1170}}{3} \right] \leq .1. \)

20. \( g(x) = x^{n(n-1)} \) is convex. Hence, by Jensen’s Inequality

\[
E[Y^{n(n-1)}] \geq E[Y]^{n(n-1)} \quad \text{Now set } Y = X^{n-1} \text{ and so}
E[X^n] \geq (E[X^{n-1}])^{n(n-1)} \text{ or } (E[X^n])^{1/n} \geq (E[X^{n-1}])^{1/(n-1)}
\]

21. No

22. (a) \( 20/26 \approx .769 \)

(b) \( 20/(20 + 36) = 5/14 \approx .357 \)

(d) \( p \approx P\{Z \geq (25.5 - 20)/\sqrt{20} \} \approx P\{Z \geq 1.23\} \approx .1093 \)

(e) \( p = .112184 \)