Real Analysis Problems

transcribed from the originals
by
William J. DeMeo

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1. Let \( f_n \) be a sequence of continuous, real valued functions on \([0, 1]\) which converges uniformly to \( f \). Prove that \( \lim_{n \to \infty} f_n(x_n) = f(1/2) \) for any sequence \( \{x_n\} \) which converges to 1/2.

b. Must the conclusion still hold if the convergence is only point-wise? Explain.

2. Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable and assume there is no \( x \in \mathbb{R} \) such that \( f(x) = f'(x) = 0 \). Show that \( S = \{x \mid 0 \leq x \leq 1, f(x) = 0\} \) is finite.

3. If \((X, \Sigma, \mu)\) is a measure space and if \( f \) is \( \mu \) integrable, show that for every \( \epsilon > 0 \) there is \( E \in \Sigma \) such that \( \mu(E) < \infty \) and

\[
\int_{X \setminus E} |f| \, d\mu < \epsilon.
\]

4. If \((X, \Sigma, \mu)\) is a measure space, \( f \) is a non-negative measurable function, and \( \nu(E) = \int_E f \, d\mu \), show that \( \nu \) is a measure.

5. Suppose \( f \) is a bounded, real valued function on \([0, 1]\). Show that \( f \) is Lebesgue measurable if and only if

\[
\sup \int \psi \, dm = \inf \int \phi \, dm
\]

where \( m \) is Lebesgue measure on \([0, 1]\), and \( \psi \) and \( \phi \) range over all simple functions, \( \psi \leq f \leq \phi \).

6.\(^1\) If \( f \) is Lebesgue integrable on \([0, 1]\) and \( \epsilon > 0 \), show that there is \( \delta > 0 \) such that for all measurable sets \( E \subset [0, 1] \) with \( m(E) < \delta \),

\[
\left| \int_E f \, dm \right| < \epsilon.
\]

7.\(^2\) Suppose \( f \) is a bounded, real valued, measurable function on \([0, 1]\) such that \( \int x^n f \, dm = 0 \) for \( n = 0, 1, 2, \ldots \), with \( m \) Lebesgue measure. Show that \( f(x) = 0 \) a.e.

8.\(^3\) If \( \mu \) and \( \nu \) are finite measures on the measurable space \((X, \Sigma)\), show that there is a nonnegative measurable function \( f \) on \( X \) such that for all \( E \) in \( \Sigma \),

\[
\int_E (1 - f) \, d\mu = \int_E f \, d\nu. \tag{1}
\]

\(^1\)See also: April ’92 (4), November ’97 (6), April 2003 (4).

\(^2\)This question has appeared often in varying forms of difficulty; cf. November ’92 (7b, easy version), November ’96 (B2, easy), November ’91 (this question, easy-moderate), April ’92 (6, moderate), November ’95 (6, hard–impossible?). I (wjd) have yet to solve the November ’95 version. I’ve seen one attempt (appearing in a notebook circulating among the grad students) which seems to assume \( f \in L^1 \), but that assumption makes the problem even easier than the others.

\(^3\)See also: November ’97 (7).
If $f$ and $g$ are integrable functions on $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$, respectively, and $F(x, y) = f(x)g(y)$, show that $F$ is integrable on $X \times Y$ and

$$\int F \, d(\mu \times \nu) = \int f \, d\mu \int g \, d\nu.$$
2 1998 April 3

Instructions Do at least four problems in Part A, and at least two problems in Part B.

PART A

1. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers, and for each positive $n$ define

$$
\hat{x}_n = \sup_{k \geq n} x_k
$$

(a) Explain why the limit $\ell = \lim_{n \to \infty} \hat{x}_n$ exists.

(b) Prove that, for any $\epsilon > 0$ and positive integer $N$, there exists an integer $k$ such that $k \geq N$ and $|x_k - \ell| < \epsilon$.

2. Let $C$ be a collection of subsets of the real line $\mathbb{R}$, and define

$$
A_{\sigma}(C) = \bigcap \{A : C \subset A \text{ and } A \text{ is a } \sigma\text{-algebra of subsets of } \mathbb{R}\}.
$$

(a) Prove that $A_{\sigma}(C)$ is a $\sigma$-algebra, that $C \subset A_{\sigma}(C)$, and that $A_{\sigma}(C) \subset A$ for any other $\sigma$-algebra $A$ containing all the sets of $C$.

(b) Let $O$ be the collection of all finite open intervals in $\mathbb{R}$, and $F$ the collection of all finite closed intervals in $\mathbb{R}$. Show that $A_{\sigma}(O) = A_{\sigma}(F)$.

3. Let $(X, A, \mu)$ be a measure space, and suppose $X = \bigcup_{n} X_n$, where $\{X_n\}_{n=1}^{\infty}$ is a pairwise disjoint collection of measurable subsets of $X$. Use the monotone convergence theorem and linearity of the integral to prove that, if $f$ is a non-negative measurable real-valued function on $X$,

$$
\int_X f \, d\mu = \sum_n \int_{X_n} f \, d\mu.
$$

4. Using the Fubini/Tonelli theorems to justify all steps, evaluate the integral

$$
\int_0^1 \int_y^1 x^{-3/2} \cos \left( \frac{\pi y}{2x} \right) \, dx \, dy.
$$

5. Let $I$ be the interval $[0,1]$, and let $C(I)$, $C(I \times I)$ denote the spaces of real valued continuous functions on $I$ and $I \times I$, respectively, with the usual supremum norm on these spaces. Show that the collection of finite sums of the form

$$
f(x, y) = \sum_i \phi_i(x)\psi_i(y),
$$

where $\phi_i, \psi_i \in C(I)$ for each $i$, is dense in $C(I \times I)$.
6. Let $m$ be Lebesgue measure on the real line $\mathbb{R}$, and for each Lebesgue measurable subset $E$ of $\mathbb{R}$ define

$$
\mu(E) = \int_E \frac{1}{1 + x^2} \, dm(x).
$$

Show that $m$ is absolutely continuous with respect to $\mu$, and compute the Radon-Nikodym derivative $dm/d\mu$.

**PART B**

7. Let $\phi(x, y) = x^2y$ be defined on the square $S = [0, 1] \times [0, 1]$ in the plane, and let $m$ be two-dimensional Lebesgue measure on $S$. Given a Borel subset $E$ of the real line $\mathbb{R}$, define

$$
\mu(E) = m(\phi^{-1}(E)).
$$

(a) Show that $\mu$ is a Borel measure on $\mathbb{R}$.

(b) Let $\chi_E$ denote the characteristic function of the set $E$. Show that

$$
\int_{\mathbb{R}} \chi_E \, d\mu = \int_S \chi_E \circ \phi \, dm.
$$

(c) Evaluate the integral

$$
\int_{-\infty}^{\infty} t^2 \, d\mu(t).
$$

8. Let $f$ be a real valued and increasing function on the real line $\mathbb{R}$, such that $f(-\infty) = 0$ and $f(\infty) = 1$. Prove that $f$ is absolutely continuous on every closed finite interval if and only if

$$
\int_{\mathbb{R}} f' \, dm = 1.
$$

9. Let $F$ be a continuous linear functional on the space $L^1[-1, 1]$, with the property that $F(f) = 0$ for all odd functions $f$ in $L^1[-1, 1]$. Show that there exists an even function $\phi$ such that

$$
F(f) = \int_{-1}^{1} f(x)\phi(x) \, dx, \quad \text{for all } f \in L^1[-1, 1].
$$

[Hint: One possible approach is to use the fact that any function in $L^p[-1, 1]$ is the sum of an odd function and an even function.]
3 2000 November 17

Do as many problems as you can. Complete solutions to five problems would be considered a good performance.

1. a. State the Inverse Function Theorem.

b. Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is an invertible linear map and that $g : \mathbb{R}^3 \to \mathbb{R}^3$ has continuous first order partial derivatives and satisfies $\|g(x)\| \leq C\|x\|^2$ for some constant $C$ and all $x \in \mathbb{R}^3$. Here $\|x\|$ denotes the usual Euclidean norm on $\mathbb{R}^3$. Prove that $f(x) = L(x) + g(x)$ is locally invertible near 0.

2. Let $f$ be a differentiable real valued function on the interval $(0,1)$, and suppose the derivative of $f$ is bounded on this interval. Prove the existence of the limit $L = \lim_{x \to 0^+} f(x)$.

3. Let $f$ and $g$ be Lebesgue integrable functions on $[0,1]$, and let $F$ and $G$ be the integrals

$$F(x) = \int_0^x f(t) \, dt, \quad G(x) = \int_0^x g(t) \, dt.$$ 

Use Fubini’s and/or Tonelli’s Theorem to prove that

$$\int_0^1 F(x)g(x) \, dx = F(1)G(1) - \int_0^1 f(x)G(x) \, dx.$$ 

Other approaches to this problem are possible, but credit will be given only to solutions based on these theorems.

4. Let $(X,A,\mu)$ be a finite measure space and suppose $\nu$ is a finite measure on $(X,A)$ that is absolutely continuous with respect to $\mu$. Prove that the norm of the Radon-Nikodym derivative $f = \frac{d\nu}{d\mu}$ is the same in $L^\infty(\mu)$ as it is in $L^\infty(\nu)$.

5. Suppose that $\{f_n\}$ is a sequence of Lebesgue measurable functions on $[0,1]$ such that $\lim_{n \to \infty} \int_0^1 |f_n| \, dx = 0$ and there is an integrable function $g$ on $[0,1]$ such that $|f_n|^2 \leq g$, for each $n$. Prove that $\lim_{n \to \infty} \int_0^1 |f_n|^2 \, dx = 0$.

6. Denote by $\mathcal{P}_e$ the family of all even polynomials. Thus a polynomial $p$ belongs to $\mathcal{P}_e$ if and only if $p(x) = \frac{p(x) + p(-x)}{2}$ for all $x$. Determine, with proof, the closure of $\mathcal{P}_e$ in $L^1[-1,1]$. You may use without proof the fact that continuous functions on $[-1,1]$ are dense in $L^1[-1,1]$.

7. Suppose that $f$ is real valued and integrable with respect to Lebesgue measure $m$ on $\mathbb{R}$ and that there are real numbers $a < b$ such that

$$a \cdot m(U) \leq \int_U f \, dm \leq b \cdot m(U),$$

for all open sets $U$ in $\mathbb{R}$. Prove that $a \leq f(x) \leq b$ a.e.
4 2001 November 26

Instructions Masters students do any 4 problems Ph.D. students do any 5 problems. Use a separate sheet of paper for each new problem.

1. Let \( \{f_n\} \) be a sequence of Lebesgue measurable functions on a set \( E \subset \mathbb{R} \), where \( E \) is of finite Lebesgue measure. Suppose that there is \( M > 0 \) such that \( |f_n(x)| \leq M \) for \( n \geq 1 \) and for all \( x \in E \), and suppose that \( \lim_{n \to \infty} f_n(x) = f(x) \) for each \( x \in E \). Use Egoroff’s theorem to prove that
\[
\int_E f(x) \, dx = \lim_{n \to \infty} \int_E f_n(x) \, dx.
\]

2. Let \( f(x) \) be a real-valued Lebesgue integrable function on \([0, 1]\).
   a. Prove that if \( f > 0 \) on a set \( F \subset [0, 1] \) of positive measure, then
   \[
   \int_F f(x) \, dx > 0.
   \]
   b. Prove that if
   \[
   \int_0^x f(x) \, dx = 0, \quad \text{for each } x \in [0, 1],
   \]
   then \( f(x) = 0 \) for almost all \( x \in [0, 1] \).

3. State each of the following:
   a. The Stone-Weierstrass Theorem
   b. The Lebesgue (dominated) Convergence Theorem
   c. Hölder’s inequality
   d. The Riesz Representation Theorem for \( L^p \)
   e. The Hahn-Banach Theorem.

4. a. State the Baire Category Theorem.
   b. Prove the following special case of the Uniform Boundedness Theorem: Let \( X \) be a (nonempty) complete metric space and let \( F \subseteq C(X) \). Suppose that for each \( x \in X \) there is a nonnegative constant \( M_x \) such that
   \[
   |f(x)| \leq M_x \quad \text{for all } f \in F.
   \]
   Prove that there is a nonempty open set \( G \subseteq X \) and a constant \( M > 0 \) such that
   \[
   |f(x)| \leq M \quad \text{holds for all } x \in G \text{ and for all } f \in F.
   \]
5. Prove or disprove:
   a. $L^2$ convergence implies pointwise convergence.
   b. 
   \[
   \lim_{n \to \infty} \int_0^\infty \frac{\sin(x^n)}{x^n} \, dx = 0.
   \]

6. Let $\{f_n\}$ be a sequence of measurable functions defined on $[0, \infty)$. If $f_n \to 0$ uniformly on $[0, \infty)$, as $n \to \infty$, then

   \[
   \lim \int_{[0, \infty)} f_n(x) \, dx = \int_{[0, \infty)} \lim f_n(x) \, dx.
   \]

7. Let $f : H \to H$ be a bounded linear functional on a separable Hilbert space $H$ (with inner product denoted by $\langle \cdot, \cdot \rangle$). Prove that there is a unique element $y \in H$ such that

   \[
   f(x) = \langle x, y \rangle \quad \text{for all } x \in H \quad \text{and} \quad \|f\| = \|y\|.
   \]

   **Hint.** You may use the following facts: A separable Hilbert space, $H$, contains a complete orthonormal sequence, $\{\phi_k\}_{k=1}^{\infty}$, satisfying the following properties: (1) If $x, y \in H$ and if $\langle x, \phi_k \rangle = \langle y, \phi_k \rangle$ for all $k$, then $x = y$. (2) Parseval’s equality holds; that is, for all $x \in H$, $\langle x, x \rangle = \sum_{k=1}^{\infty} a_k^2$, where $a_k = \langle x, \phi_k \rangle$.

8. Let $X$ be a normed linear space and let $Y$ be a Banach space. Let

   \[
   B(X, Y) = \{ A \mid A : X \to Y \text{ is a bounded linear operator} \}.
   \]

   Then with the norm $\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$, $B(X, Y)$ is a normed linear space (you need *not* show this). Prove that $B(X, Y)$ is a Banach space; that is, prove that $B(X, Y)$ is complete.
5 2007 November 16

Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^n$ is $n$-dimensional Euclidean space. Denote by $m$ Lebesgue measure on $\mathbb{R}$ and $m_n$ $n$-dimensional Lebesgue measure. Be sure to give a complete statement of any theorems from analysis that you use in your proofs below.

1. Let $\mu$ be a positive measure on a measure space $X$. Assume that $E_1, E_2, \ldots$ are measurable subsets of $X$ with the property that for $n \neq m, \mu(E_n \cap E_m) = 0$. Let $E$ be the union of these sets. Prove that
   \[ \mu(E) = \sum_{n=1}^{\infty} \mu(E_n) \]

2. (a) State a theorem that illustrates Littlewood’s Principle for pointwise a.e. convergence of a sequence of functions on $\mathbb{R}$.
   (b) Suppose that $f_n \in L^1(m)$ for $n = 1, 2, \ldots$. Assuming that $||f_n - f||_1 \to 0$ and $f_n \to g$ a.e. as $n \to \infty$, what relation exists between $f$ and $g$? Make a conjecture and then prove it using the statement in Part (a).

3. Let $K$ be a compact subset in $\mathbb{R}^3$ and let $f(x) = \text{dist}(x, K)$.
   (a) Prove that $f$ is a continuous function and that $f(x) = 0$ if and only if $x \in K$.
   (b) Let $g = \max(1 - f, 0)$ and prove that $\lim_{n \to \infty} \int\int\int g^n$ exists and is equal to $m_3(K)$.

4. Let $E$ be a Borel subset of $\mathbb{R}^2$.
   (a) Explain what this means.
   (b) Suppose that for every real number $t$ the set $E_t = \{(x, y) \in E \mid x = t\}$ is finite. Prove that $E$ is a Lebesgue null set.

5. Let $\mu$ and $\nu$ be finite positive measures on the measurable space $(X, A)$ such that $\nu \ll \mu \ll \nu$, and let $\frac{d\nu}{d(\mu + \nu)}$ represent the Radon-Nikodym derivative of $\nu$ with respect to $\mu + \nu$. Show that
   \[ 0 < \frac{d\nu}{d(\mu + \nu)} < 1 \quad \text{a.e. } [\mu]. \]

6.\footnote{On the original exam this question asked only about the special case $p = q = 2$. The question appearing here is the more general version asked at the re-write stage of the exam.} Suppose that $1 < p < \infty$ and that $q = p/(p-1)$.
   (a) Let $a_1, a_2, \ldots$ be a sequence of real numbers for which the series $\sum a_nb_n$ converges for all real sequences $\{b_n\}$ satisfying the condition $\sum |b_n|^q < \infty$. Prove that $\sum |a_n|^p < \infty$.
   (b) Discuss the cases of $p = 1$ and $p = \infty$. Prove your assertions.