A BASIC QUESTION ABOUT SELF-NORMALIZING SUBGROUPS

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[Please see the document SelfNormalizingSubgroups.pdf for an updated version of these notes.]

Suppose \( H \) and \( K \) are incomparable subgroups of a group \( G \). If their intersection is self-normalizing, must they both be self-normalizing? In other words, does the following hold?

\[
N_G(H \cap K) = H \cap K \implies N_G(H) = H \text{ and } N_G(K) = K
\]

This seemed unlikely, but a proof was not obvious to me, and counter-examples were surprisingly rare. In fact, there are no groups of order \(|G| < 96\), and no nonsolvable groups of order \(|G| < 360\), containing subgroups which violate (1).

I used GAP to find a counter-example to (1) in the alternating group on six letters. Let \( G = A_6 \), \( H = (C_3 \times C_3) : C_2 \), and \( K = S_4 \).

Then \( H \) is strictly contained in its normalizer, \( N_G(H) = (C_3 \times C_3) : C_4 \), while \( H \cap K = S_3 \) which is self-normalizing in \( A_6 \). See the Hasse diagram below which is actually the whole interval above \( S_3 \) in the subgroup lattice \( \text{Sub}[A_6] \). The GAP code verifying these assertions is given below.

Incidentally, it is easy to prove that the converse of (1) is false; that is, self-normalizing is not a property that is closed under intersection. (In fact, I believe I can prove that for any non-normal subgroup \( X \), there exists a conjugate \( X^g \) such that \( X \cap X^g \) is not self-normalizing.)

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Naturally, the next question to ask is whether \( N_G(H \cap K) = H \cap K \) implies that either \( N_G(H) = H \) or \( N_G(K) = K \). I asked mathoverflow for a counter-example; i.e. a group \( G \) with two incomparable subgroups \( H \) and \( K \) such that \( N_G(H) \leq H \), \( N_G(K) \leq K \), and \( N_G(H \cap K) = H \cap K \). Tim Dokchitser replied with the following example: Let \( G = S_3 \times S_3 \times S_3 \) be the subgroup of \( S_9 \) generated by \( (1,2,3),(1,2),(4,5,6),(4,5),(7,8,9),(7,8) \), and let \( U \) be the diagonal subgroup of \( G \); i.e., \( U = \{(x,x,x) : x \in G \} \), and \( U \) is generated by \( (1,2,3)(4,5,6)(7,8,9) \) and \( (1,2)(4,5)(7,8) \). Then let

\[
H = \langle U, (1,2,3) \rangle \text{ and } K = \langle U, (4,5,6) \rangle.
\]

Then \( U = H \cap K \) is self-normalizing, but

\[
H \cong N_G(H) \text{ and } K \cong N_G(K).
\]

This is verified in GAP with the following commands:

\begin{verbatim}
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\end{verbatim}
Figure 1. Hasse diagram of the interval $[S_3, A_6]$ in Sub$[A_6]$ drawn by the XGAP program.
GAP code

G := SmallGroup(360,118);;
StructureDescription(G); # A6

ccs := ConjugacyClassesSubgroups(G);;

H := Representative(ccs[16]);;
StructureDescription(H); # (C3 x C3) : C2
StructureDescription(Normalizer(G,H)); # (C3 x C3) : C4
    # (H is not self-normalizing)

K := Representative(ccs[17]);;
StructureDescription(K); # S4
StructureDescription(Normalizer(G,K)); # S4
    # (K is self-normalizing)

HcapK := Intersection(H,K);;
StructureDescription(HcapK); # S3
StructureDescription(Normalizer(G,HcapK)); # S3
    # (HcapK is self-normalizing)

# The filter above HcapK is:
filterHcapK := IntermediateSubgroups(G,HcapK);;
Size(filterHcapK.subgroups); # There are four intermediate subgroups:
    filterHcapK.subgroups[1]; # H = (C3 x C3) : C2
    filterHcapK.subgroups[2]; # K = S4 = N(K)
    filterHcapK.subgroups[3]; # N(H) = (C3 x C3) : C4
    filterHcapK.subgroups[4]; # A5

# Using XGAP, we can view this interval in the Hasse diagram of Sub[A6] as follows:
L := GraphicSubgroupLattice(G);
InsertVertex(L, HcapK);
InsertVertex(L, H);
InsertVertex(L, K);
InsertVertex(L, filterHcapK.subgroups[3]);
InsertVertex(L, filterHcapK.subgroups[4]);

# The same could have been achieved more efficiently with:
for k in [1..Size(filterHcapK.subgroups)] do
    InsertVertex(L, filterHcapK.subgroups[k]);
od;

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