such that $G_1, \ldots, G_n$ are non-trivial indecomposable normal subgroups of $G$. Suppose that no two of the groups $G_1, \ldots, G_n$ are isomorphic. Then $G_1, \ldots, G_n$ are characteristic subgroups of $G$ and

$$\text{Aut } G \cong \text{Aut } G_1 \times \cdots \times \text{Aut } G_n.$$  

(cf. 342; also see 94.)

437 Give an example of a finite abelian group $G$ such that $G = A \times B$, where $A$ and $B$ are non-isomorphic non-trivial indecomposable subgroups of $G$, and such that $G$ has subgroups $A^*$ and $B^*$, distinct from $A$ and $B$, and with $G = A^* \times B^*$ (cf. 8.18).

We shall now prove a result about subgroups of the direct product of two groups. In chapter 9 we shall apply this result to the extension problem: see 9.28.

8.19 Theorem (Remak [a80], Klein, Frick [b26]). Let $H$ and $K$ be normal subgroups of $G$ such that $G = H \times K$, and let $\pi$ and $\rho$ be the corresponding projections of $G$ onto $H$ and $K$, respectively. Let $L \subseteq G$. Then

(i) $(H \cap L) \trianglelefteq L \pi \trianglelefteq H, (K \cap L) \trianglelefteq L \rho \trianglelefteq K$ and $L \pi / (H \cap L) \cong L \rho / (K \cap L)$.

(ii) $L = (H \cap L) \times (K \cap L)$ if and only if $L \pi = H \cap L$ (or if and only if $L \rho = K \cap L$).

Proof. (i) We know that $\pi$ and $\rho$ are homomorphisms (3.11).

Since $H \subseteq G, (H \cap L) \subseteq L \subseteq G$. Therefore (87)

$$(H \cap L) \pi \subseteq L \pi \subseteq G \pi = H.$$  

By definition, $\pi|_H$ is the identity map on $H$.

Therefore $$(H \cap L) \pi = H \cap L.$$  

Hence $$(H \cap L) \subseteq L \pi \subseteq H.$$  

Similarly $$(K \cap L) \subseteq L \rho \subseteq K.$$  

We now define a map

$$\varphi : L \pi \rightarrow L \rho / (K \cap L).$$

For each element $h \in L \pi$, there is an element $k \in K$ such that $hk \in L$. Then $k \in L \rho$, and we define

$$h \varphi = k(K \cap L).$$

The element $k$ is not necessarily uniquely determined by $h$, and so we must check that this definition of $h \varphi$ does not depend on the choice of $k$. If also $k' \in K$ with $hk' \in L$ then

$$k^{-1}k' = (hk)^{-1}(hk') \in K \cap L,$$

and so $k'(K \cap L) = k(K \cap L)$.

Thus $\varphi$ is well defined.
Let $h_1, h_2 \in L\pi$ and let $k_1, k_2 \in K$ with $h_1 k_1, h_2 k_2 \in L$. Then $h_1 h_2 \in L\pi$, $k_1 k_2 \in K$ and, since $[H, K] = 1$,

$$(h_1 h_2)(k_1 k_2) = (h_1 k_1)(h_2 k_2) \in L.$$ 

Therefore

$$(h_1 h_2)\varphi = k_1 k_2 (K \cap L) = (h_1 \varphi)(h_2 \varphi).$$

Thus $\varphi$ is a homomorphism. It is surjective because, for any $k \in L\rho$, there is an element $h \in H$ such that $hk \in L$, and then $h \in L\pi$ and $h\varphi = k(K \cap L)$. Moreover,

$$\text{Ker } \varphi = \{h \in L\pi : hk \in L \text{ for some element } k \in K \cap L\}$$

$$= \{h \in L\pi : h \in L\}$$

$$= H \cap L \text{ (since } (H \cap L)\pi = H \cap L).$$

Therefore, by the fundamental theorem on homomorphisms,

$$L\pi/(H \cap L) = L\pi/\text{Ker } \varphi \cong \text{Im } \varphi = L\rho/(K \cap L).$$

(ii) Clearly

$$(H \cap L) \times (K \cap L) \leq L \leq L\pi \times L\rho.$$ 

If $L\pi = H \cap L$ then it follows from (i) that $L\rho = K \cap L$. Then the inclusions above imply that

$$L = (H \cap L) \times (K \cap L).$$

If, conversely, $L = (H \cap L) \times (K \cap L)$ then it is clear from the definitions of $\pi$ and $\rho$ that

$$L\pi = H \cap L \text{ and } L\rho = K \cap L.$$

8.20 Corollary. Let $G = H \times K$. Suppose that $G$ is finite and that $(|H|, |K|) = 1$. Then, for every subgroup $L$ of $G$,

$$L = (H \cap L) \times (K \cap L).$$

Proof. Let $L \leq G$ and let $\pi, \rho$ be defined as in 8.19. Then $L\pi \leq H$ and $L\rho \leq K$. Hence, by hypothesis,

$$(|L\pi|, |L\rho|) = 1.$$ 

Since, by 8.19(i), $L\pi/(H \cap L) \cong L\rho/(K \cap L)$, this implies that $|L\pi/(H \cap L)| = 1$, hence that $L\pi = H \cap L$. Thus, by 8.19(ii),

$$L = (H \cap L) \times (K \cap L).$$

Remark. This result would of course fail in general without the condition that $(|H|, |K|) = 1$. For instance, let $G = \langle a \rangle \times \langle b \rangle$ with $o(a) = o(b) = 2$. Then $\langle ab \rangle$ is a subgroup of $G$ of order 2, but $\langle a \rangle \cap \langle ab \rangle = 1 = \langle b \rangle \cap \langle ab \rangle$. 
438 Let $H$ and $K$ be normal subgroups of $G$ such that $G = H \times K$, and let $\pi$ and $\rho$ be the corresponding projections of $G$ onto $H$ and $K$, respectively. Suppose that

$$H_2 \trianglelefteq H_1 \trianglelefteq H, \quad K_2 \trianglelefteq K_1 \trianglelefteq K \quad \text{and} \quad H_1/H_2 \cong K_1/K_2.$$ 

Let $\theta$ be any isomorphism of $H_1/H_2$ onto $K_1/K_2$, and let

$$L = \{hk : h \in H_1, k \in K_1 \text{ and } (hH_2)\theta = kK_2\}.$$ 

Then $L \trianglelefteq G$ and

$$H \cap L = H_2, \quad L\pi = H_1, \quad K \cap L = K_2, \quad L\rho = K_1.$$ 

439 Let $H$ and $K$ be normal subgroups of $G$ such that $G = H \times K$, and let $\pi$ be the corresponding projection of $G$ onto $H$. Let $L \trianglelefteq G$ and let $J = (H \cap L) \times (K \cap L)$. Then $J \trianglelefteq L$ and

$$L/J \cong L\pi/(H \cap L).$$

(See 8.19. Hint. Let $\pi_1 : L \to L\pi$ be defined by restriction of $\pi$, and let $\nu : L\pi \to L\pi/(H \cap L)$ be the natural homomorphism. Consider the map $\pi_1 \nu$.)

440 (Remark [a80].) Let $H$ and $K$ be normal subgroups of $G$ such that $G = H \times K$, and let $\pi$ and $\rho$ be the corresponding projections of $G$ onto $H$ and $K$, respectively. Let $L \trianglelefteq G$. Then the following two statements are equivalent:

(i) $L \trianglelefteq G$.

(ii) $(H \cap L) \trianglelefteq H, \quad (K \cap L) \trianglelefteq K, \quad L\pi/(H \cap L) \leq Z(H/(H \cap L))$ and $L\rho/(K \cap L) \leq Z(K/(K \cap L)).$

(Hint. To prove that (ii) $\Rightarrow$ (i), let $J = (H \cap L) \times (K \cap L)$. Note that $J \trianglelefteq G$ and use 151 to show that $L/J \leq Z(G/J)$.)

441 Let $H$ and $K$ be normal subgroups of $G$ such that $G = H \times K$, and let $\pi$ and $\rho$ be the corresponding projections of $G$ onto $H$ and $K$, respectively. A subgroup $L$ of $G$ is said to be a subdirect product of $H$ and $K$ if $L\pi = H$ and $L\rho = K$.

(i) Let $L \trianglelefteq G$. Then $L$ is a subdirect product of $H$ and $K$ if and only if $HL = G = KL$.

(ii) Let $L$ be a subdirect product of $H$ and $K$. Then $L \trianglelefteq G$ if and only if $G' \trianglelefteq L$. (Hint. Apply 165 and 440.)

(iii) Suppose that $G$ is finite and that $(|H/H'|, |K/K'|) = 1$. Then no proper normal subgroup of $G$ is a subdirect product of $H$ and $K$. (Hint. Apply (i) and 8.19.)

442 Let $H \trianglelefteq G$ and $K \trianglelefteq G$. Verify that the homomorphism $\psi$ defined in 109 maps $G/(H \cap K)$ onto a subdirect product of $G/H$ and $G/K$ (see 441).

443 Let $H$ and $K$ be normal subgroups of $G$ such that $G = H \times K$. Then the following two statements are equivalent:

(i) $L$ is a subdirect product of $H$ and $K$ (441).

(ii) For some group $J$, there are surjective homomorphisms $\varphi : H \to J$ and $\psi : K \to J$ such that

$$L = \{hk : h \in H, k \in K \text{ and } h\varphi = k\psi \}.$$ 

(Hint. To prove that (i) $\Rightarrow$ (ii), see the proof of 8.19.)

It is convenient to regard the direct product of a finite number of copies of a group $G$ as a group of maps from a suitable set into $G$. We introduce this group of maps here; we shall return to it in chapter 9. The definition can also be generalized to arbitrary direct products: see 444, 445.