1. Introduction. — A theorem of Frobenius\(^1\) states that if \(G\) is a finite group, \(H\) is a subgroup of \(G\) which is its own normalizer and has trivial intersection with each of its conjugates, then the set of elements of \(G\) which do not lie in any conjugate of \(H\), together with the identity element, forms a normal subgroup \(N\) of \(G\), and an element \(h \in H, h \neq 1\), induces an automorphism of \(N\) which leaves only the identity element fixed. Conversely, if a group \(N\) possesses a fixed-point-free automorphism \(\sigma\) of prime order, then the holomorph (split extension) of \(N\) by \(\{\sigma\}\) is a group \(G\) with \(\{\sigma\}\) in the role of \(H\). Hence, groups \(N\) which can arise in Frobenius’ theorem are precisely those groups with fixed-point-free automorphisms of prime order.\(^2\)

Frobenius’ theorem left unanswered the more detailed analysis of the possible groups \(H\) and \(N\) which can arise in this manner. Burnside\(^3\) showed that the Sylow subgroups of \(H\) must be cyclic or generalized quaternion, and incorrectly stated that \(H\) must be nilpotent, an error first pointed out by Zassenhaus and discussed later by Sah.\(^4\) More recently, Suzuki\(^5\) has given a complete classification of all finite groups with cyclic Sylow subgroups for all odd primes and with 2-Sylow subgroups which are cyclic, generalized quaternion, or dihedral, so the structure of \(H\) is known.

The more detailed study of \(N\) leads to the result that if \(N\) is solvable, then it is even nilpotent. This fact seems to have been known for quite some time, but recently G. Higman\(^6\) published a proof, together with the result that if \(N\) is nilpotent, then its class of nilpotency is bounded in terms of the least prime divisor of the order of \(H\). The main object of this paper is to show that \(N\) must be nilpotent, by using Theorem A, which will be proved in a later paper. This result cannot be said to classify the structure of \(N\) in the same sense in which Suzuki classified the structure of \(H\), since it leaves unanswered which nilpotent groups may play the role of \(N\). The complete determination of the structure of \(N\) is reduced to finding all nilpotent groups possessing a fixed-point-free automorphism of prime order.

A second result of this paper concerns itself with the maximal subgroups of a finite group. Schmidt\(^7\) and Iwasawa\(^8\) independently showed that if every proper subgroup of the finite group \(G\) is nilpotent, then \(G\) is solvable. Recently Huppert\(^9\) showed that if just one of the maximal subgroups \(M\) of \(G\) is nilpotent, and if in addition the Sylow subgroups of \(M\) are regular in the sense of P. Hall,\(^10\) then \(G\) is solvable. Huppert’s theorem specializes to the case that \(M\) is abelian, a result obtained independently by Herstein,\(^11\) and with an adjustment in case \(M\) is of even order, also specializes to a result of Deskins,\(^12\) which allows \(M\) to be nilpotent of class at most two. The theorem proved here states that if \(M\) is of odd order, then \(G\) is solvable, and again the proof assumes Theorem A. Using a theorem of Brauer,\(^13\) we shall prove in a later paper that if \(G\) is not solvable and if the 2-Sylow subgroup of \(M\) is generalized quaternion or dihedral, then the structure of \(G\) is determined. \(G\) pos-
serves a normal series $G \geq G_0 > T \geq 1$ such that $[G:G_0] \leq 2$, $T$ is nilpotent, and $G_0/T \cong LF(2, q)$, the linear fractional group in two variables over $GF(q)$, where $q$ is a prime-power of the form $2^n \pm 1$.

As indicated above, the main tool in the proof of these results is Theorem A, which is correctly considered as a partial generalization of the following theorem:

A group $G$ has a factor group isomorphic to a $p$-Sylow subgroup $P$ of $G$ if, and only if, for every (normal) subgroup $Q$ of $P$ an element of order prime to $p$ which normalizes $Q$ also centralizes $Q$.

If "normal" is omitted this becomes Theorem 14.4.7 in M. Hall. If "normal" is included, it becomes a special case of Theorem A, provided $p$ is an odd prime.

**Theorem A.** Let $G$ be a finite group with a $p$-Sylow subgroup $P$, $p$ an odd prime, and let $\mathfrak{A}$ be a group of automorphisms of $G$ which leaves $P$ invariant. Suppose for every $\mathfrak{A}$-invariant normal subgroup $Q$ of $P$, elements of order prime to $p$ which normalize $Q$ also centralize $Q$. Then $G$ possesses a normal $p$-complement.

If we take $\mathfrak{A}$ to be the inner automorphisms of $G$ by the elements of $P$ we obtain the special case mentioned above. The groups $LF(2, p), p = 2^n \pm 1 > 7$ show that Theorem A is not true for $p = 2$, a fact noticed by Ito and Suzuki.

Theorem A can be considered as a nonsimplicity criterion, and in this light is analogous to results of Wielandt and P. Hall which give sufficient conditions for a group $G$ to possess a normal subgroup of index $p$. However, Theorem A (or rather its proof) can be viewed otherwise, as an attempt to analyze the manner in which the Sylow subgroups are imbedded in $G$. Such an analysis ultimately depends on studying the groups $P \cap Q$, $P$ and $Q$ being Sylow subgroups of $G$. The effectiveness of this study here depends largely on Theorem B of Hall and Higman.

2. **Consequences of Theorem A.**—Assuming Theorem A to be true, we proceed to the proof of the results indicated in the introduction.

**Theorem 1.** Let $G$ be a finite group with a fixed-point-free automorphism of prime order. Then $G$ is nilpotent.

**Proof:** As remarked earlier, it is enough to show that $G$ is solvable. We proceed by induction on $\sigma(G)$. Let $\sigma$ be the automorphism in question. Then $\sigma(x)x^{-1} = 1$ implies that $x = 1$, since $\sigma$ is fixed-point-free. Hence, if $\sigma(x)x^{-1} = \sigma(y)y^{-1}$, we have $\sigma(y^{-1}x) = y^{-1}x$, so that $x = y$. This shows that every element in $G$ has a unique representation $\sigma(x)x^{-1}$, for some $x \in G$, since $\sigma(x)x^{-1}$ sweeps out the elements of $G$ as $x$ does.

If $G$ is a 2-group, we are done. If not, let $p$ be an odd prime divisor of $\sigma(G)$, and let $P$ be a $p$-Sylow subgroup of $G$. Then $\sigma(P) = aPa^{-1}$ for some $a \in G$, since any two $p$-Sylow subgroups of $G$ are conjugate in $G$. But $a = \sigma(x)x^{-1}$ for suitable $x \in G$, so we find $\sigma(x^{-1}P) = x^{-1}Px$. Hence, without loss of generality, we suppose that $P$ is invariant under $\sigma$. If $H \neq 1$ is any $\sigma$-invariant normal subgroup of $P$, then $N_G(H)$ is also $\sigma$-invariant.

**Case 1:** $P$ possesses some $\sigma$-invariant normal subgroup $H \neq 1$, such that $N_G(H) = G$. Consider $G/H$, on which $\sigma$ induces an automorphism, say $\delta$. Suppose $\delta(ah) = aH$, or equivalently, $\sigma(a)a^{-1} \in H$. Since $H$ is $\sigma$-invariant, every element of $H$ has a unique representation in the form $\sigma(h)h^{-1}$ with $h \in H$. Hence $\sigma(a)a^{-1} = \sigma(h)h^{-1}$, from which we conclude that $a = h \in H$. In other words, $\sigma$ induces a fixed-point-free automorphism on $G/H$. By induction, $G/H$ is solvable, so that $G$, being an extension of a solvable group by a solvable group, is solvable.
Case 2: If $H \neq 1$ is any $\sigma$-invariant normal subgroup of $P$, then $N_G(H) \lessdot G$. Since $H$ is $\sigma$-invariant, so is $N_G(H)$. Since $N_G(H) \lessdot G$, by induction $N_G(H)$ is nilpotent, so elements of order prime to $p$ which normalize $H$ centralize $H$. This says that $G$, with $\{ \sigma \}$ in the role of $\mathfrak{A}$, and $P$ as the $p$-Sylow subgroup, satisfies the hypotheses of Theorem A, so that $G$ possesses a normal $p$-complement $K$, which is necessarily characteristic, hence $\sigma$-invariant. By induction, $K$ is nilpotent, so that $G$, being an extension of a nilpotent group by a nilpotent group, is solvable.

**Theorem 2.** Let $G$ be a finite group, and suppose that one of the maximal subgroups $M$ of $G$ is nilpotent of odd order. Then $G$ is solvable.

**Proof:** By induction, we can assume that if $H \neq 1$ is any normal subgroup of $M$, then

$$N_G(H) = M.$$  \hfill (1)

Let $P$ be a $p$-Sylow subgroup of $M$, $p \mid o(M)$. $P$ is normal in $M$ since $M$ is nilpotent. Hence, $N_G(P) = M$, by (1). If $P$ were not a $p$-Sylow subgroup of $G$, then $P$ would be contained as a normal subgroup of index $p$ in a $p$-group $\bar{P}$, which gives $\bar{P} \leq N_G(P) = M$, contrary to our choice of $P$ as a $p$-Sylow subgroup of $M$. Hence, $P$ is a $p$-Sylow subgroup of $G$. Since $M$ is nilpotent, any normal subgroup of $P$ is normal in $M$. Let $\mathfrak{A}$ be the group of inner automorphisms of $G$ by the elements of $P$. We see that $G$, $\mathfrak{A}$ and $P$ satisfy the hypotheses of Theorem A. Hence $G$ has $p$-complement, $K(P)$. The intersection of the $K(P)$ for $P$ ranging over the Sylow subgroups of $M$ is thus a normal complement for $M$, say $K$.

Since $M$ is nilpotent, $Z(M) \neq 1$. An element $x \in Z(M), x \neq 1$, $x$ of prime order, induces an automorphism of $K$. This must be a fixed-point-free automorphism of $K$, otherwise $C_G(x)$ contains $M$ as a proper subgroup, so that $C_G(x) = G$ by maximality of $M$, which contradicts (1). Hence $K$ is nilpotent by Theorem 1, so that $G$, being an extension of a nilpotent group by a nilpotent group is solvable.

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PRODUCTION OF STERILITY IN MICE BY DEUTERIUM OXIDE*

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Deuterium oxide in the drinking water of either male or female mice produces sterility. Male mice are more sensitive to the effects of D₂O than are female mice, and C₅₇ mice appear to be more sensitive than Swiss mice.

We have investigated further some of the conditions—with particular reference to time—of D₂O treatment required to produce sterile C₅₇ male mice. These experiments indicate that the sensitive phase of sperm production is probably centered around the late prophase of meiosis.

Experimental Procedures and Results.—Production of sterility: In order to determine the minimum time required to produce sterility in C₅₇ male mice by the substitution of D₂O in the drinking water for ordinary water, 10 male mice maintained on 30 per cent D₂O were individually mated for one-week intervals with C₅₇ female mice. The female mice were maintained on 30 per cent D₂O only for the one-week period in which they were caged with a male mouse. At the end of each week, the females were replaced by additional normal female mice, and the female mice that had been removed were caged individually to determine if pregnant. Offspring were counted two weeks after birth, and those pairs which did not produce a litter during a 28-day period after the initial mating were considered to be sterile. The results summarized in Table 1 indicate that all male mice were sterile 28 days after initiation of D₂O treatment. The male mice may have been sterile as soon as 23 days after the initiation of D₂O treatment, since eight of the pairs that were first mated at 21 days produced litters 20 to 22 days after the start of mating.

In order to determine if treatment with D₂O for a relatively short time would produce sterility in mice subsequent to withdrawal of the D₂O, groups of 10 C₅₇ male mice were given 30 per cent D₂O for periods of one, two, and three weeks and individually mated for one-week intervals with C₅₇ female mice. No subsequent sterility was observed in the group of 10 mice treated with D₂O for one week. The mice treated for two weeks with 30 per cent D₂O were not sterile at the end of the two-week treatment or one week later, but they were sterile four weeks after the initiation of treatment, i.e., two weeks after D₂O treatment had been discontinued. (Seven pairs produced no litters and litters from the other three died. Five of the 10 mice mated were sterile 35 days from the initiation of treatment or 21 days after D₂O treatment had been terminated, and the offspring from the two other litters