In this note we list all congruence lattices of transitive G-sets in \( \text{Eq}(n) \) for \( n = 3, \ldots, 15 \) (with a few for \( n = 16 \) as well). This is possible because of the following observation: If \( G \) is an arbitrary transitive permutation group of degree \( n \) (the number of moved points), then the index of the stabilizer \( G_x \) is \( [G : G_x] = n \), and the transitive G-set \( \langle X; G \rangle \cong \langle G/G_x; G \rangle \) has \( |X| = n \) and congruence lattice \( \text{Con}(X; G) \leq \text{Eq}(n) \), which is isomorphic to the interval \([G_x, G]\) in the subgroup lattice of \( G \). Thus, sublattices of \( \text{Eq}(n) \) which are congruence lattices of transitive G-sets are the intervals above stabilizer subgroups of transitive groups of degree \( n \).

GAP has a library of transitive permutation groups of degree at most 30. Therefore, for a transitive G-set \( \langle X; G \rangle \) with \(|X| \leq 30\), the shape of \( \text{Con}(X; G) \) can be computed with three simple GAP commands. Take, for example, \( G = \text{TransitiveGroup}(4,2) \) (the second transitive group of degree 4). The covering relations of the sublattice \( \text{Con}(X; G) \leq \text{Eq}(4) \) are found by

\[
gap> \text{G := TransitiveGroup}(4,2); \quad \% \text{return } E(4) = 2[x]2 \\
gap> \text{H := Stabilizer(G,1); \quad \% \text{return } Group(()) \\
gap> \text{intHG := IntermediateSubgroups(G,H); \\
rec( \text{subgroups := [ Group([ (1,2)(3,4) ]), Group([ (1,4)(2,3) ]), Group([ (1,3)(2,4) ] ) ]}, \text{inclusions := [ [ 0, 1 ], [ 0, 2 ], [ 0, 3 ], [ 1, 4 ], [ 2, 4 ], [ 3, 4 ] ] })
\]

The list \( \text{intHG.inclusions} \) (the last line above) shows that \( \text{Con}(X; G) \cong M_3 \).

The following displays the number of transitive permutation groups of degree at most 20:

\[
gap> \text{List([1..20], x->NrTransitiveGroups(x));} \\
[ 1, 1, 2, 5, 5, 16, 7, 50, 34, 45, 8, 301, 9, 63, 104, 1954, 10, 983, 8, 1117 ]
\]

Many of these groups are primitive, that is, the congruence lattice of the associated G-set is just the two element lattice. For example, all transitive groups of prime degree are primitive. So, in the list below, we only present those G-set congruence lattices in \( \text{Eq}(n) \) for \( n = 4, 6, 8, 9, 10, 12, 14, 15 \).

Properties of transitive groups a given degree, \( n \), can be checked with the \text{AllTransitiveGroups} function with \text{NrMovedPoints} parameter set to \( n \). For example, we can check for primitivity of transitive groups of degrees 5, 6, 7, and 11 as follows:

\[
gap> \text{List(AllTransitiveGroups(NrMovedPoints,5),IsPrimitive);} \\
[ true, true, true, true, true ]
\]

\[
gap> \text{List(AllTransitiveGroups(NrMovedPoints,6),IsPrimitive);} \\
[ false, false, false, false, false, false, false, false, false, false, false, false, false, false, false, false, false, false, false, false, true,... \\
\ldots false, true, true, true ]
\]

\[
gap> \text{List(AllTransitiveGroups(NrMovedPoints,7),IsPrimitive);} \\
[ true, true, true, true, true, true, true, true ]
\]

\[
gap> \text{List(AllTransitiveGroups(NrMovedPoints,11),IsPrimitive);} \\
[ true, true, true, true, true, true, true, true, true, true ]
\]

William DeMeo <williamdemeo@gmail.com>
Figure 1: Transitive G-set congruence lattices in Eq(4)

\[
\begin{align*}
[1, \mathbb{C}_4] & \quad \text{TransitiveGroup(4,1)} \\
[1, \mathbb{C}_2 \times \mathbb{C}_2] & \quad \text{TransitiveGroup(4,2)}
\end{align*}
\]

Figure 2: Transitive G-set congruence lattices in Eq(6)

\[
\begin{align*}
[1, \mathbb{C}_6] & \quad \text{TransitiveGroup(6,1)} \\
[1, \mathbb{S}_3] & \quad \text{TransitiveGroup(6,2)} \\
[\mathbb{C}_2, \mathbb{A}_4] & \quad \text{TransitiveGroup(6,4)}
\end{align*}
\]
Figure 3: Transitive G-set congruence lattices in Eq(8)
Figure 4: Transitive G-set congruence lattices in Eq(9)

[1, C9]  [1, C3 x C3]  [C2, C3 x S3]
TransitiveGroup(9,1)  TransitiveGroup(9,2)  TransitiveGroup(9,4)

Figure 5: Transitive G-set congruence lattices in Eq(10)

[1, C10]  [1, D10]  [C5, C5 x D10]
TransitiveGroup(10,1)  TransitiveGroup(10,2)  TransitiveGroup(10,6)
Figure 6: Transitive G-set congruence lattices in Eq(12)
Figure 7: Transitive G-set congruence lattices in Eq(12) (continued)

[C3, S3 × S3] TransitiveGroup(12,16)
[C3, (C3 × C3) : C4] TransitiveGroup(12,17)
[C3, C6 × S3] TransitiveGroup(12,18)
[C3, C3 × (C3 : C4)] TransitiveGroup(12,19)
[C3, C3 × A4] TransitiveGroup(12,20)
[C2 × C2, C2 × S4] TransitiveGroup(12,22)
[C2 × C2, C4 × A4] TransitiveGroup(12,29)
[C2 × C2, A4 : C4] TransitiveGroup(12,30)
[C5, A5] TransitiveGroup(12,33)
Figure 8: Transitive G-set congruence lattices in Eq(12) (continued)

\[ \text{TransitiveGroup}(12,47) \]

Figure 9: Transitive G-set congruence lattices in Eq(14)

\[ \text{TransitiveGroup}(14,1) \]
\[ \text{TransitiveGroup}(14,2) \]
\[ \text{TransitiveGroup}(14,6) \]

Figure 10: Transitive G-set congruence lattices in Eq(15)

\[ \text{TransitiveGroup}(15,1) \]
\[ \text{TransitiveGroup}(15,5) \]
Figure 11: Transitive G-set congruence lattices in Eq(16)
Figure 12: Transitive G-set congruence lattices in Eq(16) (continued)

- $[\mathbb{C}^3, \text{SL}(2,3) : \mathbb{C}^2]$ TransitiveGroup(16,60)
- $[\mathbb{C}^3, (\mathbb{C}^4 \times \mathbb{C}^4) : \mathbb{C}^3]$ TransitiveGroup(16,63)
- $[\mathbb{C}^3, (\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2) : \mathbb{C}^3]$ TransitiveGroup(16,64)
- $[\mathbb{C}^2 \times \mathbb{C}^2, ((\mathbb{C}^8 \times \mathbb{C}^2) : \mathbb{C}^2) : \mathbb{C}^2]$ TransitiveGroup(16,84)
- $[\mathbb{S}^3, ((\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2) : \mathbb{C}^3) : \mathbb{C}^2]$ TransitiveGroup(16,194)
- $[\mathbb{C}^7, \mathbb{C}^2 \times ((\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2) : \mathbb{C}^7)]$ TransitiveGroup(16,196)
- $[\mathbb{A}^4, \mathbb{C}^2 \times (((\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2) : \mathbb{C}^3) : \mathbb{C}^2)]$ TransitiveGroup(16,416)
- $[\mathbb{A}^4, (((\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2) : \mathbb{C}^3) : \mathbb{C}^2) : \mathbb{C}^2]$ TransitiveGroup(16,417)
- $[\mathbb{A}^4, ((\mathbb{C}^2 \times ((\mathbb{C}^4 \times \mathbb{C}^2) : \mathbb{C}^2)) : \mathbb{C}^2) : \mathbb{C}^2]$ TransitiveGroup(16,418)
Figure 13: Transitive G-set congruence lattices in Eq(16) (continued)

\[ A_4, \left(\left(\left(C_2 \times D_8\right) : C_2\right) : C_3\right) : C_2 \]

\[ \text{TransitiveGroup(16,425)} \]

\[ A_4, \left(\left(\left(C_4 \times C_4\right) : C_2\right) : C_2\right) : C_3 \]

\[ \text{TransitiveGroup(16,437)} \]