The finite congruence lattice problem

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0. Quick introduction

1. Reductions (today)
   - minimal unary algebras
   - transitive permutation groups
   - almost simple groups
   - twisted wreath products

2. Background (Wednesday)
   - more details
   - some history

3. Some constructions (Thursday)
   - closure properties
   - hereditary congruence lattices
0. Quick introduction

**Theorem** (Grätzer György – Schmidt Tamás, 1963) For every algebraic lattice \( L \) there exists an algebra with congruence lattice isomorphic to \( L \).

\( L \) is **representable** (as a congruence lattice)

Proofs by Grätzer and Schmidt (1963), Lampe (1973), Pudlák (1976), Tůma (1989) (almost) always yield an infinite algebra, even if \( L \) is finite.

**The finite congruence lattice problem**
Is it true that for every finite lattice \( L \) there exists a finite algebra with congruence lattice isomorphic to \( L \) ?

\( L \) is **finitely representable** (as a congruence lattice)
1. Reductions

\( \text{Con}(U; F) = \text{Con}(U; \text{Pol}_1(U; F)) \), so we will assume that the algebra is unary, and the operations form a transformation monoid.

If \( L \) is finitely representable, we will take a representation where \( |U| \) is minimal such that \( \text{Con}(U; F) \cong L \).

**Theorem** (Pavel Pudlák – P³, 1980)

If \( L \) satisfies certain assumptions, then the operations form a transitive permutation group.
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**Theorem** (Pavel Pudlák – P³, 1980)

If \( L \) satisfies certain assumptions (that will be specified in Lecture 2), then (in the minimal unary representation of \( L \)) the operations form a transitive permutation group (after removing the constant operations).
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This leads to the following equivalent formulation of the finite congruence representation problem:

Is it true that for every finite lattice \( L \) there exists a finite group \( G \) and a (core-free) subgroup \( H \leq G \) such that the interval \( \text{Int}(H; G) \) of the subgroup lattice consisting of the subgroups containing \( H \) is isomorphic to \( L \)?
Transitive permutation groups

$G$ a group acting from the right on the set $U$

$(U; G)$ is also called a **$G$-set**

Notation: $u \mapsto u^g$ ($u \in U$, $g \in G$)

$u^{(g_1g_2)} = (u^{g_1})^{g_2}$, $u^1 = u$

**stabilizer** of $u \in U$: $G_u = \{g \in G | u^g = u\} \leq G$

$G_{u^g} = g^{-1}G_{ug}$

$G$ is **transitive**: $\forall u, v \in U \exists g \in G : u^g = v$, i.e., the unary algebra $(U; G)$ has no proper subalgebra.
The core

The kernel of a transitive action of $G$ on $U$ is

$$\{g \in G \mid \forall v \in U : v^g = v\} = \bigcap_{v \in U} G_v = \bigcap_{g \in G} g^{-1}G_u g,$$

the largest normal subgroup of $G$ contained in the stabilizer $G_u$, the core of $G_u$.

So we can assume that $H$ is core-free in $G$, i.e., $\bigcap_{g \in G} g^{-1}Hg = 1$.

In fact, if $N \triangleleft G$ and $N \leq H$, then $\text{Int}(H; G) \cong \text{Int}(H/N; G/N)$. 
The strategy

Is it true that for every finite lattice $L$ there exists a finite group $G$ and a core-free subgroup $H \leq G$ such that $\text{Int}(H; G) \cong L$?

We try to reduce the question to the case when $G$ is an **almost simple group**: $G$ has a normal subgroup $S$ which is a nonabelian simple group and $C_G(S) = 1$. Hence $G$ embeds into $\text{Aut}(S)$. If we identify $S$ with the subgroup of $\text{Aut}(S)$ consisting of the inner automorphisms (the conjugations by elements of $S$), then we obtain $\text{Inn}(S) \leq G \leq \text{Aut}(S)$.

Fact (**Schreier’s Conjecture**): For every finite simple group $S$, the **outer automorphism group** $\text{Aut}(S)/\text{Inn}(S)$ is solvable. Established using the Classification of Finite Simple Groups (**CFSG**).

If the problem is reduced to the case of almost simple groups, then using the CFSG one can attack it by a case-by-case analysis.
Three important papers


Their conclusion: $G$ is almost simple
Three important papers


Their conclusion: $G$ is almost simple or a twisted wreath product.
What to do now?

Analyze the case of twisted wreath products.

Either show that such groups cannot represent all finite lattices, so get a reduction to the almost simple case,

or represent every finite lattice as an interval in the subgroup lattice of a twisted wreath product, perhaps in some “combinatorial” way.
Twisted wreath product (Bernhard H. Neumann, 1963)

Ingredients:

- base group $B$,
- outer group $H$,
- a subgroup $A \leq H$,
- a homomorphism $\alpha : A \rightarrow \text{Aut}(B)$; it defines an action of $A$ on $B$, which will be denoted — as before — by $b^a$ (instead of $b^{\alpha(a)}$).

(If $\alpha$ maps every element of $A$ to the identical automorphism of $B$, then we obtain the ordinary wreath product — without twist.)
Twisted wreath product (1)

Given: $H \geq A \to \text{Aut}(B)$

Construction:

$B^H = \{ f : H \to B \} \ (\text{all functions}).$ It is a group with pointwise multiplication, isomorphic to $B^{|H|} = B \times \cdots \times B.$

Define the action of $H$ on $B^H$ by

$$f^h(x) = f(hx) \quad (f \in B^H, h \in H, x \in H).$$

It is indeed an action:

$f^{h_1 h_2}(x) = f((h_1 h_2)x) = f(h_1(h_2x)) = f^{h_1}(h_2x) = (f^{h_1})^{h_2}(x).$

$f \mapsto f^h \ (\text{for a fixed } h \in H)$ is an automorphism of $B^H$:

$(f_1 f_2)^h(x) = (f_1 f_2)(hx) = f_1(hx) f_2(hx) = f_1^h(x) f_2^h(x).$

(The semidirect product of $H$ and $B^H$ is the regular wreath product of $B$ and $H.$)
Twisted wreath product (2)

Given: $H \geq A \rightarrow \text{Aut}(B)$.
So far we have constructed $B^H$ and the action of $H$ on it.
Here comes the twist:

Let

$$U = \{ u : H \rightarrow B \mid \forall x \in H, a \in A : u(xa) = u(x)^a \}.$$ 

It is a subgroup of $B^H$, and $U \cong B^{|H:A|}$. Namely, the value $u(x)$ determines the values on the whole left coset $xA$.

If $u \in U$, $h \in H$, then $u^h(xa) = u(hxa) = u(hx)^a = (u^h(x))^a$, so $u^h \in U$, i.e., $U$ is an $H$-invariant subgroup of $B^H$.

$HU$ is the **twisted wreath product** of the ingredients $(B, H, A, \alpha)$.

$$(h_1u_1)(h_2u_2) = (h_1h_2)(u_1^{h_2}u_2)$$
The interval $\text{Int}(H; HU)$

If $H \leq X \leq HU$, then $X = H(U \cap X)$, where $U \cap X$ is an $H$-invariant subgroup of $U$.

Conversely, if $V \leq U$ is $H$-invariant, then $H \leq HV \leq HU$.

So

$\text{Int}(H; HU) \cong \text{Sub}^H(U),$

the lattice of $H$-invariant subgroups of $U$. 
Restrictive conditions

In general, $\text{Sub}^H(U)$ is too complex, but the reduction in the papers of Baddeley, Börner, and Aschbacher leads to twisted wreath products with severely restricted ingredients.

- (a) $B$ is a nonabelian simple group,
- (b) $\alpha(A) \geq \text{Inn}(B)$,
- (c) $\text{Ker} \alpha$ is core-free in $H$.

We have to determine $\text{Sub}^H(U)$, the lattice of $H$-invariant subgroups of $U$ under these hypotheses.
Sub\(^H(U)\) (1)

Let \(1 \neq V \leq U \leq B^H\) be a nontrivial \(H\)-invariant subgroup.

Let \(V(x) = \{v(x) \mid v \in V\} \leq B\ (x \in H)\).

Since \(V\) is \(H\)-invariant,

\[
V(x) = \{v(x1) \mid v \in V\} = \{v^x(1) \mid v \in V\} = V(1),
\]

so \(V(x)\) is independent of \(x\).

For \(a \in A\),

\[
V(1) = V(a) = \{v(1a) \mid v \in V\} = \{v(1)^a \mid v \in V\} = V(1)^a.
\]

Now since every inner automorphism of \(B\) is induced by some element of \(A\) (Condition (b)), \(V(1)\) is a normal subgroup of \(B\), hence by the simplicity of \(B\) (Condition (a)), \(V(x) = V(1) = B\), i.e., \(V\) is a subdirect power of \(B\).
Subdirect powers

What does a subdirect power of a nonabelian simple group look like?

(It is an essential ingredient in the proof of the O’Nan–Scott[–Aschbacher] Theorem on primitive permutation groups.)

**Lemma.** Let $B$ be a nonabelian simple group and $V \leq B^n$ a subdirect power of $B$. Then $V$ is isomorphic to $B^m$ for some $1 \leq m \leq n$ via an isomorphism $B^m \rightarrow V$,

$$(b_1, \ldots, b_m) \mapsto (b_{i(1)}^{\beta_1}, \ldots, b_{i(n)}^{\beta_n}),$$

where $i : \{1, \ldots, n\} \rightarrow \{1, \ldots, m\}$ is a surjective map and $\beta_1, \ldots, \beta_n \in \text{Aut}(B)$.

**Example.** $n = 5$, $m = 2$:

$V = \{(b_1, b_1^\beta, b_2, b_1^\gamma, b_2^\beta) \mid b_1, b_2 \in B\} \leq B^5$
Let $1 \neq V \leq U \leq B^H$ be a nontrivial $H$-invariant subgroup.

Define $T = \{ t \in H \mid \forall v \in V : v(1) = 1 \implies v(t) = 1 \}$, and for $t \in T$ let $\beta(t) \in \text{Aut}(B)$ such that $v(t) = v(1)^{\beta(t)}$.

If $u \in U$, then $u(a) = u(1)^a$, hence $A \leq T$, and $\beta(a) = \alpha(a)$ for all $a \in A$.

$v(xt) = v^x(t) = v^x(1)^{\beta(t)} = v(x)^{\beta(t)}$ ($x \in H$, $t \in T$),
$v(t_1 t_2) = v(t_1)^{\beta(t_2)} = v(1)^{\beta(t_1)\beta(t_2)}$ ($t_1, t_2 \in T$), so $T$ is a subgroup and $\beta(t_1 t_2) = \beta(t_1)\beta(t_2)$, i.e., $\beta : T \to \text{Aut}(B)$ is a homomorphism.

Thus $HV$ is the twisted wreath product constructed from the data $(B, H, T, \beta)$.

**Theorem.** The dual of the lattice $\text{Sub}^H(U) \cong \text{Int}(H; HU)$ is isomorphic to the lattice of all extensions of $\alpha$ to subgroups of $H$ with a largest element added.
Examples

\[ B = A_5, \ H = S_5 \times A_5, \ A = \{(a, a) \mid a \in A_5\}, \ \alpha \ \text{the natural mapping} \ A \cong A_5 \to \Aut(B) = \Aut(A_5) \cong S_5 \]

The subgroups containing \( A \) are
\[ A < A_5 \times A_5 < S_5 \times A_5 = H. \]
There are two extensions of \( \alpha \) to both \( A_5 \times A_5 \) and \( S_5 \times A_5 \) (the projections).
So \( \Int(H; HU) \) is the hexagon.

Aschbacher gave a somewhat different example yielding the hexagon. It also provided an answer to a question about von Neumann algebras left open by Watatani (1996).
\[ B = A_5, \ H = A_6 \times A_6, \ A = \{(a, a) \mid a \in A_5\} \]
The subgroups containing \( A \) are
\[ A < A_5 \times A_5 < A_6 \times A_5, \ A_5 \times A_6 < A_6 \times A_6 = H. \]
There are two extensions of \( \alpha \) to \( A_5 \times A_5 \), unique extensions to both \( A_6 \times A_5 \) and \( A_5 \times A_6 \), and no extension to \( H = A_6 \times A_6 \).
Happy birthday

I learned about the finite congruence lattice problem at Ervin Fried’s seminar in 1976.

Ervin Fried was born on September 6, 1929.

Happy birthday!