The finite congruence lattice problem
3. Some constructions

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Partition lattices

Obviously, the partition lattice $\text{Part}(k)$ (the lattice of all equivalence relations on a $k$-element set) is a congruence lattice.

It is also an interval in a subgroup lattice, for example

$$\text{Int}(S_1 \times S_2 \times S_4 \times \cdots \times S_{2^{k-1}}; S_{2^{k-1}}) \cong \text{Part}(k).$$

**Lemma.** Let $S$ be a finite nonabelian simple group, and $D = \{(s, s, \ldots, s) | s \in S\}$ the diagonal subgroup in $S^k$. Then $\text{Int}(D; S^k)$ is the dual of $\text{Part}(k)$.

**Proof.** Every subgroup $D \leq X \leq S^k$ is a subdirect power of the form $\{(s_{i(1)}^{\alpha_1}, s_{i(2)}^{\alpha_2}, \ldots, s_{i(k)}^{\alpha_k}) | s_1, \ldots, s_m \in S\}$, where $i : \{1, \ldots, k\} \to \{1, \ldots, m\}$ and $\alpha_1, \ldots, \alpha_k \in \text{Aut}(S)$. Since $D \leq X$, all automorphisms can be taken to the identity.

For example $X = \{(s_1, s_2, s_1, s_3, s_2) | s_1, s_2, s_3 \in S\}$. So $X$ is determined by the kernel of the mapping $i$.

The larger the kernel of $i$ is, the smaller is the corresponding subgroup.
The dual lattice

**Theorem** (Kurzweil, 1985; Netter)
The dual of a finitely representable lattice is also finitely representable.
Proof. Let $L \cong \text{Con}(U; F)$ for a unary algebra $(U; F)$. Take any finite nonabelian simple group $S$.
Take the permutation group (unary algebra) $(S^U : D; S^U)$, its congruence lattice is the dual of $\text{Part}(U)$.
The elements of $S^U$ are functions $U \to S$, the elements of the diagonal subgroup $D$ are the constant functions.
The operations $f \in F$, $f : U \to U$ give rise to operations on $S^U$ simply by composition: if $g : U \to S$, then $f(g) : U \to S$ is defined by $(f(g))(u) = g(f(u))$.
If we multiply $g$ by a constant, then $f(g)$ will be multiplied by the same constant, therefore $f$ can be defined on $S^U : D$ as well.
A congruence of $(S^U : D; S^U)$ remains a congruence of the algebra $(S^U : D; S^U \cup F)$ iff it corresponds to a partition invariant under all $f \in F$, that is, iff it is a congruence of $(U; F)$.
Intervals and sublattices

If $\vartheta \in \text{Con}(U; F)$, then $\text{Con}(U/\vartheta; F) \cong \text{Int}(\vartheta; 1)$, so a filter in the congruence lattice is again a congruence lattice.

The theorem about the representation of the dual lattice then yields:

**Corollary.** Every interval is a finitely representable lattice is also finitely representable.

John Snow (2000) gave a direct proof.

Is every sublattice of a finitely representable lattice also finitely representable?

**Theorem** (Pudlák and Tůma, 1980)
Every finite lattice can be embedded into a suitable finite partition lattice.
Is every homomorphic image of a finitely representable lattice also finitely representable?

**Lemma** *(P^5, 1980)* Let \( e \in \text{Pol}_1(U; F) \) be an idempotent function \( (e^2 = e) \), then the restriction is a lattice homomorphism of \( \text{Con}(U; F) \) onto \( \text{Con}(e(U); eF) \) (the **induced algebra**).

**Lemma.** The direct product of finitely representable lattices is also finitely representable.

Proof. Take the product of transformation monoids containing all constants, then 
\[
\text{Con}(U_1 \times U_2; F_1 \times F_2) = \text{Con}(U_1; F_1) \times \text{Con}(U_2; F_2).
\]
Here \((f_1, f_2)(u_1, u_2) = (f_1(u_1), f_2(u_2))\).
Lemma 1 (Snow, 2000) Let $\alpha, \beta \in \text{Con}(U; F)$. Then we can find additional operations $F^*$ so that

$$\text{Con}(U; F \cup F^*) = \{ \gamma \in \text{Con}(U; F) | \gamma \leq \alpha \text{ or } \gamma \geq \beta \}.$$ 

Proof. Let $F^*$ consist of those unary operations whose kernel contains $\alpha$ and the image lies in one $\beta$-class.

If $\alpha \geq \gamma \in \text{Con}(U; F)$, $f^* \in F^*$ and $(u, v) \in \gamma \leq \alpha$, then $f^*(u) = f^*(v)$, so $f^*$ preserves $\gamma$.

If $\beta \leq \gamma \in \text{Con}(U; F)$, $f^* \in F^*$ (and $(u, v) \in \gamma$), then $f^*(u)$ and $f^*(v)$ lie in the same $\beta$-class, so in the same $\gamma$-class, hence $f^*$ preserves $\gamma$.

If $\gamma \in \text{Con}(U; F)$ is such that $\alpha \not\geq \gamma$ and $\beta \not\leq \gamma$, then choose $(u, v) \in \gamma \setminus \alpha$ and $(u', v') \in \beta \setminus \gamma$. Let $f^*$ take the value $u'$ on the $\alpha$-class of $u$ and $v'$ everywhere else. Then $f^* \in F^*$, $(u, v) \in \gamma$, but $(f^*(u), f^*(v)) = (u', v') \notin \gamma$. 


**Lemma 2** (Snow) Let $\beta_1 \leq \alpha_1$, $\beta_2 \leq \alpha_2$ be congruences of $(U; F)$ such that $\beta_1 \lor \beta_2 = 1$ and $\alpha_1 \land \alpha_2 = 0$. Then we can find additional operations $F^*$ so that

$$\text{Con}(U; F \cup F^*) = \{0\} \cup \text{Int}(\beta_1; \alpha_1) \cup \text{Int}(\beta_2; \alpha_2) \cup \{1\}.$$ 

Proof. Take the additional operations provided by Lemma 1 both for the pair $\beta_1$, $\alpha_2$ and for $\beta_2$, $\alpha_1$. Then the congruences that remain are those which lie

(above $\beta_1$ or below $\alpha_2$) and (above $\beta_2$ or below $\alpha_1$),

that is

$$\gamma \geq \beta_1 \lor \beta_2 \text{ or } \beta_1 \leq \gamma \leq \alpha_1 \text{ or } \beta_2 \leq \gamma \leq \alpha_2 \text{ or } \gamma \leq \alpha_1 \land \alpha_2.$$
Theorem (Snow, 2000) The ordinal sum and the parallel sum of two finitely representable lattices are also finitely representable.

Proof. The ordinal sum of $L_1$ and $L_2$ is their disjoint union, where every element of $L_1$ is smaller than each element of $L_2$. A somewhat more natural version of the ordinal sum of two lattices is obtained from the usual ordinal sum if we identify the largest element of $L_1$ with the smallest element of $L_2$. This construct will be denoted by $L_1 + L_2$.

(A noncommutative—but associative—addition!)

The usual ordinal sum of $L_1$ and $L_2$ is just $L_1 + 2 + L_2$.

Now take a finite algebra with congruence lattice $L_1 \times L_2$ and use Snowmobile-1 with $\alpha = \beta = (1, 0)$. Then we obtain a finite algebra with congruence lattice $L_1 + L_2$. 
More Snow (2)

The **parallel sum** of $L_1$ and $L_2$ is the disjoint union

$$\{0\} \cup L_1 \cup L_2 \cup \{1\},$$

where the elements of $L_1$ and $L_2$ are pairwise incomparable.

First we prove the claim when $L_2$ is the 1-element lattice, and we will denote the parallel sum of $L$ and the 1-element lattice by $L^+$. (It has three additional elements: $0 < m < 1$.)

Let $\text{Con}(U; F) \cong L$. Take the algebra $(U \times \{1, 2\}; F)$, where $f(u, i) = (f(u), i)$. Use Snowmobile-2 with the following congruences: $\alpha_1$ has two classes $U \times \{1\}$ and $U \times \{2\}$, $\beta_1$ has one nonsingleton class $U \times \{1\}$, $\alpha_2 = \beta_2$ has 2-element classes $\{(u, 1), (u, 2)\}$.

So we obtain an algebra with congruence lattice isomorphic to $L^+$.

In general, the parallel sum of $L_1$ and $L_2$ can be obtained using Snowmobile-2 in the congruence lattice $L_1^+ \times L_2^+$ with $\alpha_1 = (1_1, m)$, $\beta_1 = (0_1, m)$, $\alpha_2 = (m, 1_2)$, $\beta_2 = (m, 0_2)$. 
Some classes of finitely representable lattices

**Definition** A finite(ly generated) lattice $L$ is **lower bounded** if there exists an epimorphism $\varphi : FL(X) \to L$ such that $\forall a \in L : \{ w \in FL(X) \mid \varphi(w) \geq a \}$ has a least element.

**Theorem.** A finite lattice $L$ is lower bounded iff $L$ and $\text{Con}(L)$ has the same number of join irreducible elements.

**Theorem** (Pudlák and Tůma, 1976)
The finite lower bounded lattices are finitely representable.
(They called these lattices **finitely fermentable**.)

**Theorem** (Snow, 2000) Every finite lattice which contains no three element antichains is finitely representable.

**Theorem** (Snow, 2003) Every finite lattice in the variety generated by $M_3$ is finitely representable.
Hereditary congruence lattices

The idea of Snow’s proof is this: If \( L \) is a finite lattice in the variety generated by \( M_3 \), then \( L \) is a 0–1-sublattice of \( M_3^k \) for some \( k \). \( M_3 \cong \text{Part}(3) \), so \( L \) can be considered as a 0–1-sublattice of \( \text{Part}(3)^k \subset \text{Part}(3^k) \). He then proves that every 0–1-sublattice of \( \text{Part}(3)^k \) is the congruence lattice of some algebra on the \( 3^k \)-element set.

Definition (Hegedűs and P³, 2005) A 0–1-sublattice \( L \) of all equivalence relations on a finite set \( U \) is called a **hereditary congruence lattice** if every 0–1-sublattice \( L' \subset L \) is the congruence lattice of a suitable algebra on \( U \). Furthermore, \( L \) is called **power-hereditary** if \( L^k \) as a lattice of equivalence relations on \( U^k \) is a hereditary congruence lattice for every \( k \geq 1 \).

In this language Snow’s result says that the lattice of all equivalences on the 3-element set is power-hereditary.
Snakes

\[ \text{Con}(Z_2 \times Z_2) \cong M_3 \] is also power-hereditary (Hegedűs and P^3), but there are non-power-hereditary representations of \( M_3 \) as well (P^3, 2006).

**Problem.** Is there a hereditary congruence lattice isomorphic to \( M_4 \)? That is \( \text{Con}(U; F) \cong M_4 \) and for every nontrivial congruence \( \vartheta_i \) \( (i = 1, \ldots, 4) \) there is a unary function \( f_i^* \) such that \( \text{Con}(U, F \cup \{ f_i^* \}) = \text{Con}(U; F) \setminus \{ \vartheta_i \} \).

A **snake** of length \( n \geq 2 \) is a modular lattice glued together from \( n - 1 \) \( M_3 \)’s.

**Theorem** (Hegedűs and P^3, 2005) Every finite lattice in the variety generated by all snakes is finitely representable.

We construct operator groups \((A; +, F)\), where \((A; +)\) is an elementary abelian 2-group and \( F \) is a suitable ring of endomorphisms of \((A; +)\).