Sets

Definition: A <u>set</u> is a well-defined collection of distinct objects. The objects of a set are called its <u>elements</u>. If a set has no elements, it is called the <u>empty set</u> and is denoted by $\varnothing$.

Note: $\varnothing$ is not the same as 0.

Example: The set of digits consists of the collection of numbers $0, 1, 2, 3, 4, 5, 6, 7, 8,$ and 9. If we use the symbol D to denote the set of digits, then we can write

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

In this notation, the braces { } are used to enclose the objects in the set.

Another way to denote a set D of digits is

$$D = \{x|x \text{ is a digit}\}.$$

***The above expression is read as "D is the set of all x such that x is a digit."

Example: $E = \{x|x \text{ is an even digit}\} = \{0, 2, 4, 6, 8\}$.

Example: $O = \{x|x \text{ is an odd digit}\} = \{1, 3, 5, 7, 9\}$.

Example: $A = \{x|x \text{ is an integer and } 2 < x < 7\} = \{3, 4, 5, 6\}$.

Definition: If A and B are sets, the <u>intersection</u> of A with B, denoted by $A \bigcap B$, is the set consisting of elements that belong to both A and B.

Definition: The <u>union</u> of A with B, denoted by $A \bigcup B$, is the set consisting of elements that belong to either A or B, or both.

Example: Let $A = \{1, 3, 5, 8\}$, $B = \{3, 5, 7\}$, and $C = \{2, 4, 6, 8\}$.
Find $A \bigcap B$, $A \bigcup B$, and $B \bigcap (A \bigcup C)$.

<u>Answer:</u>

$$\begin{aligned} A \bigcap B &= \{1, 3, 5, 8\} \bigcap \{3, 5, 7\} \\ &= \{3, 5\} \end{aligned}$$

$$\begin{aligned} A \bigcup B &= \{1, 3, 5, 8\} \bigcup \{3, 5, 7\} \\ &= \{1, 3, 5, 7, 8\} \end{aligned}$$

$$
\begin{aligned}
B\bigcap(A\bigcup C) &= \{3,5,7\}\bigcap\Big[\{1,3,5,8\}\bigcup\{2,4,6,8\}\Big] \\
&= \{3,5,7\}\bigcap\{1,2,3,4,5,6,8\} \\
&= \{3,5\}
\end{aligned}
$$

Definition: The universal set is the set consisting of all the elements that we wish to consider.

Definition: If A is a set (in the universal set), the compliment of A, denoted by $\bar{A}$, is the set consisting of all the elements in the universal set that are not in A.

Example: If the universal set is $U = \{1,2,3,4,5,6,7,8,9\}$ and if $A = \{1,3,5,7,9\}$, then $\bar{A} = \{2,4,6,8\}$.

It follows that $A\bigcup\bar{A} = U$ and $A\bigcap\bar{A} = \varnothing$.

Classify Numbers

It is helpful to classify various kinds of numbers as sets. The *counting numbers* or *natural numbers*, are the numbers in the set $\mathbb{N} = \{1,2,3,4,\ldots\}$.

Definition: (Integers)**.** The integers are the set of numbers $\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

Definition: (Rational Numbers)**.** A rational number is a number that can be expressed as a quotient $\frac{a}{b}$ of two integers. The integer a is called the **numerator**, and the integer b, which cannot be 0, is called the **denominator**. The rational numbers are the numbers in the set $\mathbb{Q} = \{x|x = \frac{a}{b}$, where a, b are integers and $b \neq 0\}$.

***Note:** It is important to realize that any integer a can be expressed as $\frac{a}{1}$. Thus all integers are also rational numbers. We say the integers are a *subset* of the rational numbers: $\mathbb{Z} \subset \mathbb{Q}$.

Fact: It can be shown that every rational number may be represented by a decimal that either terminates or is nonterminating with a repeating block of digits, and vice versa.

Definition: (Irrational Numbers)**.** Irrational numbers are real numbers that cannot be written in the form $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$.

Example: The Greeks discovered that the ratio of the circumference, C, to the diameter, d, of any circle is the irrational number $\pi = \frac{C}{d}$.

Definition: (Real Numbers)**.** The set of real numbers, $\mathbb{R}$, is the union of the set of rational numbers with the set of irrational numbers.

Properties of $\mathbb{R}$

Commutative Properties: For $a, b \in \mathbb{R}$,

- $a + b = b + a$

- $a * b = b * a$

Associative Properties: For $a, b, c \in \mathbb{R}$,

- $a + (b + c) = (a + b) + c = a + b + c$

- $a * (b * c) = (a * b) * c = a * b * c$

Distributive Properties: For $a, b, c \in \mathbb{R}$,

- $a * (b + c) = a * b + a * c$

- $(a + b) * c = a * c + b * c$

Identities: For $a \in \mathbb{R}$,

- $0 + a = a + 0 = a$

- $a * 1 = 1 * a = a$

Additive Inverse: For $a \in \mathbb{R}$,

- $a + (-a) = (-a) + a = 0$

Multiplicative Inverse: For $a \neq 0 \in \mathbb{R}$,

- $a * \frac{1}{a} = \frac{1}{a} * a = 1$
  Note: The multiplicative inverse, $\frac{1}{a}$ is also known as the reciprocal of a.

Definition: (Difference)**.** The difference $a - b$ is defined as:

$$a - b = a + (-b).$$

Definition: (Quotient)**.** If $b \neq 0 \in \mathbb{R}$, the quotient $\frac{a}{b}$ is defined as:

$$\frac{a}{b} = a * \frac{1}{b}.$$

Note: Division by 0 is not defined. For example, $\frac{2}{0} = x$ means there exists an $x \in \mathbb{R}$ such that $0 * x = 2$. But $0 * x = 0$ for all $x \in \mathbb{R}$.

More Properties of $\mathbb{R}$

- $a * 0 = 0$
- $\frac{0}{a} = 0$ and $\frac{a}{a} = 1$ if $a \neq 0$
- $a(-b) = -(ab) = (-a)b$; $(-a)(-b) = ab$; $-(-a) = a$; $(-a) = -(a)$

Cancellation Properties:

- If $ac = bc$, then $a = b$ if $c \neq 0$.
- $\frac{ac}{bc} = \frac{a}{b} * \frac{c}{c} = \frac{a}{b} * 1 = \frac{a}{b}$ if $b, c \neq 0$.

Arithmetic of Quotients:

- $\frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}$ if $b, d \neq 0$.
- $\frac{a}{b} + \frac{c}{d} = \frac{a}{b} * \frac{d}{d} + \frac{c}{d} * \frac{b}{b} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad+cb}{bd}$ if $b, c \neq 0$.
- $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} * \frac{d}{c} = \frac{ad}{bc}$ if $b, c, d \neq 0$.

Class Work: Simplify/Solve the following.

1. $3x + 2x = 5$
2. $(x + 2)(x - 4)$
3. $\frac{4+0}{2*1}$
4. $\frac{1}{3} * \frac{2}{2}$
5. $\frac{\left(\frac{1}{2}\right)}{2}$
6. $-(x - 2)(x)$
7. $(-4)(3x)$
8. $\frac{3}{5} * \frac{2}{3}$

Exponents

Definition:. Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then we define $a^n = a * a * ... * a$ where there are n factors of a. If $a \neq 0$, then we can also define $a^{-n} = \frac{1}{a^n}$ and $a^0 = 1$.

Laws of Exponents: Let $a \in \mathbb{R}$, $m, n \in \mathbb{N}$.

1. $a^m * a^n = a^{m+n}$
2. $(a^m)^n = a^{m*n}$
3. $(ab)^n = (a^n)(b^n)$
4. $\frac{a^m}{a^n} = a^{m-n}$ if $a \neq 0$
5. $(\frac{a}{b})^n = \frac{a^n}{b^n}$ is $b \neq 0$

Examples:

1. $3^2 * 3 = 3^{2+1} = 3^3 = 27$
2. $(5^2)^3 = 5^6$
3. $(4x)^2 = 4^2 * x^2 = 16x^2$
4. $\frac{3^4}{3^2} = 3^{4-2} = 3^2 = 9$
5. $\left(\frac{1}{2}\right)^3 = \frac{1^3}{2^3} = \frac{1}{8}$

Square Root

Definition:. Square Root If $a \geq 0$, then a number b such that $b^2 = a$ is a square root of a. If $b > 0$ then b is the *principle square root* of a and we write $b = \sqrt{a} = a^{\frac{1}{2}}$.

Note: The square root can be a negative number. For example, since $7^2 = 49$ and $(-7)^2 = 49$, both 7 and (-7) are square roots of 49.

Observations:

1. $a < 0$ don't have any square roots in $\mathbb{R}$ since the square of any real number is always non-negative.
2. $0^2 = 0$ implies $\sqrt{0} = 0$

The Real Number Line

$\mathbb{R}$ can be represented by points on a line called the Real Number Line. An important property of the real number line follows from the fact that, given 2 numbers a and b either:

1. a is to the left of b
2. a is at the same location as b
3. a is to the right of b

We write $a < b$; $a = b$; $a > b$ respectively. Similarly, if a is "less than or equal" to b we write $a \leq b$.

Note: $a > 0$ is equivalent to a is positive and $a < 0$ is equivalent to a is negative.

Definition: (Absolute Value)**.** The absolute value, $|a|$, of a number $a \in \mathbb{R}$ is the distance from 0 to a on the number line:

Example: -4 is 4 units from 0. Thus $|-4| = 4$.

***$|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$.

Example: $-4 < 0$ so $|-4| = -(-4) = 4$.

Definition: (Distance between two points)**.** The distance between 2 points a, b on the number line is $|b - a|$

Definition: (Domain)**.** The set of values that a variable may assume is called the domain of the variable.

Example: The domain of $\frac{5}{x-2}$ is $\{x | x \neq 2\}$ since the denominator is undefined for 0.