Solving Inequalities

**Definition 1:** An open interval denoted by $(a, b)$ consists of all real numbers $x$ for which

$$a < x < b$$

An open interval denoted by $(a, \infty)$ consists of all real numbers $x$ for which

$$x > a$$

An open interval denoted by $(-\infty, b)$ consists of all real numbers $x$ for which

$$x < b$$

An open interval denoted by $(-\infty, \infty)$ consists of all real numbers $x$.

**Definition 2:** A closed interval denoted by $[a, b]$ consists of all real numbers $x$ for which

$$a \leq x \leq b$$

**Definition 3:** A half-open, or half-closed interval denoted by $(a, b]$ consists of all real numbers $x$ for which

$$a < x \leq b,$$

and a half-open denoted by $[a, b)$ consists of all real numbers $x$ for which

$$a \leq x < b.$$  

The half-open interval denoted by $[a, \infty)$ consists of all real numbers $x$ for which

$$x \geq a,$$

and the half-open interval denoted by $(\infty, b]$ consists of all real numbers $x$ for which

$$x \leq b.$$ 

**Example 1:** Write each inequality using interval notation:

- $1 \leq x \leq 3 \Rightarrow [1, 3]$
- $x > 5 \Rightarrow (5, \infty)$
- $-12 < x \leq 7 \Rightarrow (-12, 7]$
- $x \leq 1 \Rightarrow (-\infty, 1]$

**Properties of Equivalent Inequalities:**

- You may **add** or **subtract** anything from both sides of an inequality.
- You may **multiply** or **divide** both sides by a **positive** number.
- If you **multiply** or **divide** by a **negative** number, you must **change** the direction of the inequality (multiplying $6 > 3$ by $-1$ gives the correct inequality $-6 < -3$).
- If you don’t know whether a number (with unknowns) is positive or negative, you cannot multiply or divide by it!
Examples:

- $5x \leq 15 \implies x \leq 3$
- $-5x \leq 15 \implies x \geq -3$
- $x < \frac{1}{2}$ does NOT imply $x^2 < 1$ (because we don’t know if $x > 0$ or $x < 0$)

**Example 3:** Solve the inequality $3 - 2x < 5$.

\[
\begin{align*}
3 - 2x &< 5 \\
-2x &< 5 - 3 \\
-2x &< 2 \\
x &> -1
\end{align*}
\]

**Example 4:** Solve the inequality $4x + 7 \geq 2x - 3$.

\[
\begin{align*}
4x + 7 &\geq 2x - 3 \\
4x - 2x &\geq -3 - 7 \\
2x &\geq -10 \\
x &\geq -5
\end{align*}
\]

**Example 5:** Solve the combined inequality $-5 < 3x - 2 < 1$.

\[
\begin{align*}
-5 &< 3x - 2 < 1 \\
-5 + 2 &< 3x < 1 + 2 \\
-3 &< 3x < 3 \\
-1 &< x < 1
\end{align*}
\]

This is the same solution as $x \in (-1, 1)$.

**Example 6:** Solve the combined inequality $-1 \leq \frac{3 - 5x}{2} \leq 9$.

\[
\begin{align*}
-1 &\leq \frac{3 - 5x}{2} \leq 9 \\
-2 &\leq 3 - 5x \leq 18 \\
-5 &\leq -5x \leq 15 \\
1 &\geq x \geq -3
\end{align*}
\]

This is the same solution as $x \in [-3, 1]$.

**Exercises:**

1. $8 - 4(2 - x) \leq -2x$
2. $x(4x + 3) \leq (2x + 1)^2$
Solving Rational Inequalities

As with equalities, you may add or subtract anything from both sides of an inequality. You may multiply or divide both sides by a positive number. If you multiply or divide by a negative number, you must change the direction of the inequality (multiplying $6 > 3$ by $-1$ gives the correct inequality $-6 < -3$).

If you don’t know whether a number (with unknowns) is positive or negative, you cannot multiply or divide by it!

Example: $x < \frac{1}{x}$ does NOT imply $x^2 < 1$ (because we don’t know if $x > 0$ or $x < 0$)

To solve problems like $x < \frac{1}{x}$, use the key-number method:

1. Find key numbers $x$ where $f(x) = 0$ or $f(x)$ is undefined.
2. On the key intervals (intervals separated by key numbers, may include the key numbers) $f(x)$ is either $\leq 0$ or $\leq 0$. To find out which, evaluate $f(x)$ at any point inside the interval (not the key numbers). We don’t need the number, just the sign.

Example: Solve for $x : x < \frac{1}{x}$.

\[
x < \frac{1}{x} \iff x - \frac{1}{x} < 0 \iff \frac{x^2 - 1}{x} < 0 \iff \frac{(x-1)(x+1)}{x} < 0
\]

So $f(x) = \frac{(x-1)(x+1)}{x}$.

Key numbers: $x = -1, x = 0, x = 1$.

Key intervals: $(-\infty, -1), (-1, 0), (0, 1), (1, \infty)$.

Pick $-2 \in (-\infty, -1), -\frac{1}{2} \in (-1, 0), \frac{1}{2} \in (0, 1), \text{ and } 2 \in (1, \infty)$.

So $f(-2) = -, f(-\frac{1}{2}) = +, f(\frac{1}{2}) = -, f(2) = +$.

\[
\begin{array}{ccccccc}
\vline & - & + & - & 0 & - & + \\
\hline
-1 & & & & & & \\
\vline \hline
0 & & & & & & \\
\vline \hline
1 & & & & & & \\
\vline \hline
\end{array}
\]

Since we already showed, $x < \frac{1}{x} \iff f(x) < 0$, then $x < \frac{1}{x} \iff x \in (-\infty, -1) \cup (0, 1)$.

Example: Solve for $x : \frac{x-1}{x^2 - x - 2} \geq 0$.

\[
\frac{x-1}{x^2 - x - 2} \iff \frac{x-1}{(x+1)(x-2)} \geq 0
\]

So, $f(x) = \frac{x-1}{(x+1)(x-2)} \geq 0$.

Key numbers: $-1, 1, 2$. Undefined at $-1, 2$.

Key intervals: $(-\infty, -1), (-1, 1], [1, 2), (2, \infty)$

Then, $f(-2) < 0, f(0) > 0, f(\frac{3}{2}) < 0, f(3) > 0$.

\[
\begin{array}{ccccccc}
\vline & - & + & - & \frac{3}{2} & - & + \\
\hline
-1 & & & & & & \\
\vline \hline
0 & & & & & & \\
\vline \hline
2 & & & & & & \\
\vline \hline
\end{array}
\]

Therefore $f(x) = \frac{x-1}{(x+1)(x-2)} \geq 0 \iff x \in (-1, 1] \cup (2, \infty)$.
Example: Solve for $x : \frac{(x+1)(x-3)}{(x-1)^2} \geq 0$

Key numbers: $-1, 1, 3$

Test each interval. We get:

Thus, $x \in (-\infty, -1] \cup [3, \infty)$.

Example: Solve for $x : \frac{(x+1)(3-x)}{(x-1)^2(x-5)} \leq 0$

Key numbers: $-1, 1, 3, 5$

Test each interval. We get:

Thus, $x \in [-1, 1) \cup (1, 3] \cup (5, \infty)$.

Steps:
1. Move everything to left side.
2. Combine everything into one fraction.
3. Factor the numerator and denominator into linear factors.
4. Find key numbers
5. Use point test to test each key interval.
6. Write the answer by picking the right intervals.

Facts:
1. The sign (of $f(x)$) changes at the key number with odd power factor.
2. The sign (of $f(x)$) stays the same at the key number with even power factor.