Composition of Functions

**Definition:** For functions $f$ and $g$, define $f \circ g$, the composition of $f$ and $g$ by,

$$(f \circ g)(x) = f(g(x))$$

$\text{Dom}(f \circ g) = \{x \in \text{dom}(g)|g(x) \in \text{dom}(f)\}$

**Example:** Suppose $f(x) = x - 2$ and $g(x) = x^2$.

(a) Find $(f \circ g)$ and $(g \circ f)$.

$$(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 - 2$$

$$(g \circ f)(x) = g(f(x)) = g(x - 2) = (x - 2)^2 = x^2 - 4x + 4$$

***Note: $(f \circ g) \neq (g \circ f)$

(b) Find $(f \circ g)(2)$ and $(g \circ f)(2)$.

$$(f \circ g)(2) = f(g(2)) = f(4) = 4 - 2 = 2$$

$$(g \circ f)(2) = g(f(2)) = g(2 - 2) = g(0) = 0^2 = 0$$

**Example:** For $f(x) = 3x + 4$ and $g(x) = 5$, find $(f \circ g)$ and $(g \circ f)$.

$$(f \circ g)(x) = f(g(x)) = f(5) = 3(5) + 4 = 19$$

$$(g \circ f)(x) = g(3x + 4) = 5$$

**Example:** For $f$ and $g$ below, note that

$f(-3) = 1, f(-1) = 2, f(2) = 3, f(4) = 2, g(-2) = -1, g(-1) = -2, g(1) = -1, g(2) = 2$.

Find:

$$(f \circ g)(2) = f(g(2)) = f(3)$$

$$(g \circ f)(2) = g(f(2)) = g(3) = \text{undefined},$$

$$(f \circ f)(-1) = f(f(-1)) = f(2) = 3$$
Example: $f(x) = x + \frac{1}{x}$. Find $(f \circ f)$.

$$(f \circ f)(x) = f(f(x)) = (x + \frac{1}{x}) + \left(\frac{1}{x + \frac{1}{x}}\right) = x + \frac{1}{x} + \frac{x}{x^2 + 1}$$

Next we want to write a function as a composition of 2 simpler functions.

Example: Write $(x^2 + 2)^6$ as a composition $f(g(x))$.

$(x^2 + 2)^6$ has an inner function $g(x) = x^2 + 2$.

Then the outer function $f(x)$ does what remains to be done: $f(x) = x^6$.

Check: $f(g(x)) = f(x^2 + 2) = (x^2 + 2)^6$.

Example: Write $4\frac{1}{x} + 3$ as a composition $f(g(x))$.

$4\left(\frac{1}{x}\right) + 3$ as inner function $g(x) = \frac{1}{x}$.

Then the outer function $f(x)$ does what remains to be done: $f(x) = 4x + 3$.

Check: $f(g(x)) = f\left(\frac{1}{x}\right) = 4\left(\frac{1}{x}\right) + 3$.

Example: Write $\sqrt{x^2 + 1}$ as a composition $f(g(x))$.

$\sqrt{x^2 + 1}$ has inner function $g(x) = x + 1$.

So $f(x) = \sqrt{x}$.

Check: $f(g(x)) = f(x + 1) = \sqrt{x^2 + 1}$.

Example: Write $x^4 + x^2 + 1$ as a composition.

$$x^4 + x^2 + 1 = (x^2)^2 + x^2 + 1 \quad \Rightarrow \quad g(x) = x^2 \quad \Rightarrow \quad f(x) = x^2 + x + 1.$$  

Check: $f(g(x)) = f(x^2) = (x^2)^2 + x^2 + 2 = x^4 + x^2 + 1$.

Example: Write $\frac{1}{1 + |x|}$ as the composition of 3 functions $h(f(g(x)))$.

$\frac{1}{1 + |x|}$ has inner function $g(x) = |x|$.

$\frac{1}{1 + x}$ has inner function $f(x) = 1 + x$.

Thus $h(x) = \frac{1}{x}$.

Check: $h(f(g(x))) = h(f(|x|)) = h(1 + |x|) = \frac{1}{1 + |x|}$.
Inverse Functions

**Definition:** $f^{-1}$, the inverse of $f$, is the function, if any, such that

\[
(f \circ f^{-1})(x) = x \quad \text{when } f^{-1}(x) \text{ is defined and}
\]

\[
(f^{-1} \circ f)(x) = x \quad \text{when } f(x) \text{ is defined}
\]

**Example:** $f(x) = 2x$, $g(x) = \frac{x}{2}$

Consider $f(g(x)) = f\left(\frac{x}{2}\right) = 2\left(\frac{x}{2}\right) = x$ and $g(f(x)) = g(2x) = \frac{2x}{2} = x$. Thus, $g(x)$ is an inverse function of $f(x)$. I can write $f^{-1}(x) = g(x) = \frac{x}{2}$.

**Definition:** $f$ is 1-1 ("one-to-one") $\iff x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

**Example:** $f(x) = 3x$ is 1-1 but $g(x) = x^2$ is not 1-1 since $1 = -1$ but $(-1)^2 = 1^2$.

**Horizontal Line Test:** If every horizontal line intersects the graph of a function $f$ in at most one point, the $f$ is one-to-one.

**Example:** Which of the following has an inverse?

\[
\text{Answer: The first graph has an inverse and the second graph doesn’t.}
\]

**Theorem:** The function $f$ has an inverse if and only if $f$ is 1-1.

**Theorem:** $y = f^{-1}(x) \iff f(y) = x$.

To find $f^{-1}(x)$ for complicated functions:

(1) Switch $x$ and $y$ in $y = f(x)$, i.e. write $x = f(y)$.

(2) Solve for $y$. After that you can replace $y$ by $f^{-1}(x)$.

**Example:** $f(x) = x^3$, find $f^{-1}(x)$.

\[
f(y) = x \iff y^3 = x \iff y = x^{\frac{1}{3}} \iff f^{-1}(x) = x^{\frac{1}{3}}.
\]

**Note:** $f^{-1}(x) \neq (f(x))^{-1}$.

$f^{-1}(x)$ is the inverse of $f(x)$ and $(f(x))^{-1} = \frac{1}{f(x)}$ is the reciprocal of $f(x)$. 

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Example: \( f(x) = \frac{2x+1}{x-1} \), find \( f^{-1}(x) \).

\[
\begin{align*}
y &= \frac{2x+1}{x-1} \\
x &= \frac{2y+1}{y-1} \\
x(y-1) &= 2y + 1 \\
xy - x &= 2y + 1 \\
xy - 2y &= x + 1 \\
y(x-2) &= x + 1 \\
y &= \frac{x+1}{x-2}
\end{align*}
\]

Now let’s look at how the graph of \( f \) is related to the graph of \( f^{-1} \).

Since \( y = f^{-1}(x) \iff f(y) = x \). Thus the graph of \( y = f^{-1}(x) \) is the graph of \( f(y) = x \) which is just the graph of \( f(x) = y \) with \( x \) and \( y \) interchanged. Interchanging \( x \) and \( y \) reflects the plane around the major diagonal \( y = x \).

Theorem: The graph of \( y = f^{-1}(x) \) is the reflection of the graph of \( y = f(x) \) across the major diagonal \( y = x \).

Theorem: The domain of \( f^{-1} \) is the range of \( f \). The range of \( f^{-1} \) is the domain of \( f \).