Please write your solutions neatly and bind them with a staple. Unless otherwise stated, you must carefully justify everything you write. I encourage you to work in groups on these, but you must write your solutions in your own words. The exercise(s) marked with a ⋆ are optional, and they will count as a bonus if you solve them correctly. Your solutions are due at the end of class. No late assignments will be accepted.

Unless otherwise specified, $F$ is an arbitrary field and $V$ is a vector space over $F$.

1. Use the calculation techniques presented in class to answer the following questions:
   (a) Are $(X + 1)^2, X^3 - X, X + 1, X^3 + X + 1$ linearly independent in $F[X]_3$?
   (b) Is $(1, 2, 3) ∈ \text{Span}\{(2, 1, 4), (-1, 2, -3)\}$, regarded as vectors in $\mathbb{R}^3$? As vectors in $F^3$?
   (c) Recall that the set of $2 \times 2$ matrices is a vector space. Is \[
   \begin{bmatrix}
   1 & 2 \\
   3 & 4
   \end{bmatrix}
   \]
in the span of the matrices \[
   \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix} \right\}?
   \]

2. Let $\mathcal{S}t = \{(1, 0), (0, 1)\}$ (standard basis), $\mathcal{B} = \{(1, 2), (1, 3)\}$ and $\mathcal{C} = \{(1, -1), (-1, 2)\}$ be three bases of $\mathbb{R}^2$.
   (a) Find the change of basis matrices $\mathcal{S}t M_\mathcal{B}$, $\mathcal{B} M_\mathcal{C}$, and $\mathcal{C} M_\mathcal{B}$.
   (b) If $v = \{1, -1\}$, find $[v]_{\mathcal{S}t}$, $[v]_{\mathcal{B}}$, and $[v]_{\mathcal{C}}$.


5. Exercise 14, page 36 of Axler.

6. A linear form is a homogeneous linear polynomial $L(x_1, \ldots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n$ with coefficients in the field $F$. A linear form is called nontrivial if at least one of its coefficients is nonzero. If $\gamma \in F$ and $L(x_1, \ldots, x_n)$ is nontrivial, the set of solutions to the linear equation $L(x_1, \ldots, x_n) = \gamma$ is called a hyperplane in $F^n$. This equation is called homogeneous if $\gamma = 0$, and it is inhomogeneous if $\gamma \neq 0$. (We may also say that the equation $L(x_1, \ldots, x_n) = \gamma$ is nontrivial if the linear form $L(x_1, \ldots, x_n)$ is nontrivial.)

   If $L(x_1, \ldots, x_n)$ is a nontrivial linear form with coefficients in $F$, prove that the hyperplane $H = \{(\beta_1, \ldots, \beta_n) \in F^n : L(\beta_1, \ldots, \beta_n) = 0\}$ is a subspace of $F^n$ of dimension $n - 1$. 


7. If $W \subset F^n$ is a subspace and $H$ is a hyperplane, prove that $\dim(W \cap H) \geq \dim(W) - 1$, with equality if and only if $W \nsubseteq H$. (Geometrically, this means the procedure of intersecting a subspace with a hyperplane should decrease the dimension by 1. Algebraically, it means a single linear equation should impose 1 dimension worth of restriction on a subspace.)

8. Now consider a system of $m$ nontrivial linear forms in $n$ variables $x_1, \ldots, x_n$ with coefficients in the field $F$:

\[
\begin{align*}
\alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n &= 0 \\
\alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n &= 0 \\
\vdots \\
\alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \alpha_{mn}x_n &= 0 
\end{align*}
\]

Let $Z \subset F^n$ be the set of all solutions to the system $(†)$. Prove that $Z$ is the intersection of a collection of hyperplanes (see exercise 6 for the definition of a hyperplane.) Deduce that $Z$ is a subspace of $F^n$, and show that $\dim(W) \geq n - m$.

[Recall that the converse is also true; we sketched the proof in class.]

9. Calculate the dimension of the subspace $W \subset \mathbb{R}^5$ defined by the equations

\[
\begin{align*}
3x_1 + 21x_2 + 9x_4 &= 0 \\
x_1 + 7x_2 - x_3 - 2x_4 - x_5 &= 0 \\
2x_1 + 14x_2 + 6x_4 &= 0 \\
6x_1 + 42x_2 - x_3 + 13x_4 &= 0 
\end{align*}
\]

10. In this exercise, we define a way to make new vector spaces out of old ones. Let $V_1$ and $V_2$ be vector spaces over the field $F$. Define the external direct sum $V_1 \oplus V_2$ to be the set of all ordered pairs $(w_1, w_2)$ with $w_1 \in V_1$ and $w_2 \in V_2$ under the operations

\[
\begin{align*}
(w_1, w_2) + (w'_1, w'_2) &:= (w_1 + w'_1, w_2 + w'_2) \quad \text{for all } w_1, w'_1 \in V_1 \text{ and } w_2, w'_2 \in V_2 \\
\alpha.(w_1, w_2) &:= (\alpha w_1, \alpha w_2) \quad \text{for all } \alpha \in F \text{ and } w_1 \in V_1, w_2 \in V_2. 
\end{align*}
\]

Then $V_1 \oplus V_2$ is a vector space. (You should verify this, but you do not need to write it up.) Sometimes $V_1 \oplus V_2$ is called the direct sum of $V_1$ and $V_2$. You should be careful to distinguish this concept from our notion of the direct sum of subspaces. We will return to this in a later assignment to explain why the notations look so similar.

If $V_1$ and $V_2$ are finite dimensional vector spaces, prove that the external direct sum $V_1 \oplus V_2$ is also finite dimensional, and that $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$.

11. ∗ Fix a scalar $\alpha \in F$.

(a) Prove that the set $\mathcal{C} = \{1, (X - \alpha), (X - \alpha)^2, \ldots, (X - \alpha)^n\}$ is a basis of $F[X]_n$.

(b) Now let $F = \mathbb{R}$. If $\mathcal{B} = \{1, X, X^2, \ldots, X^n\}$ is the usual monomial basis, find a simple form for the change-of-basis matrices $\mathcal{B}M_\mathcal{C}$ and $\mathcal{C}M_\mathcal{B}$.