1. Let $\phi : V \to W$ be a function. Prove that $\phi$ is a linear map if and only if 
   
   $$\phi(\alpha v + v') = \alpha \phi(v) + \phi(v')$$

   for all $\alpha \in F$ and $v, v' \in V$.

   (Compare with the definition on p.38 of the text.)

2. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

   $$\phi(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, x_2 + x_3, -x_1 + 3x_2 + 4x_3).$$

   Find a basis for the kernel $\ker(\phi)$ and a basis for the image $\text{im}(\phi)$.

3. Let $\phi : V \to W$ be a linear map. Prove that $\phi$ is invertible if and only if $\phi$ is bijective.

   (Note that $V$ and $W$ are not necessarily finite dimensional.)

4. Let $\phi : V \to W$ be a linear map between finite dimensional vector spaces, and suppose

   $$\dim(V) = \dim(W).$$

   Prove that $\phi$ is invertible if and only if $\phi$ is surjective. [Hint: We proved the related statement “$\phi$ is invertible if and only if $\phi$ is injective” in class. You may want to use this fact in your proof.]

5. Let $\phi : V \to W$ be a linear map between finite dimensional vector spaces. Prove the following statements:

   (a) $\phi$ is injective if and only if \{\phi(v_1), \ldots, \phi(v_m)\} is a linearly independent list in $W$

       whenever \{v_1, \ldots, v_m\} is a linearly independent list in $V$.

   (b) $\phi$ is invertible if and only if $\phi(v_1), \ldots, \phi(v_n)$ is a basis of $W$

       whenever \{v_1, \ldots, v_n\} is a basis of $V$.

   [The hypothesis that $V$ is finite dimensional is completely unnecessary in this problem, and your proof probably works for an arbitrary vector space.]

6. Suppose $\phi$ and $\psi$ are linear operators on a finite dimensional vector space $V$. Prove that

   $$\phi \circ \psi$$

   is invertible if and only if both $\phi$ and $\psi$ are invertible, in which case $(\phi \circ \psi)^{-1} = \psi^{-1} \circ \phi^{-1}$.


8. Exercise 23, page 61 of Axler. [Note: We gave an example in class that shows this result can be false when $V$ is infinite dimensional.]
9. Let $U, W \subset V$ be subspaces such that $V = U \oplus W$. Recall that $P_{U,W} : V \to V$ is the projection to $U$ relative to $W$. Prove that we have the following equality of linear operators:

$$\text{id}_V = P_{U,W} + P_{W,U}.$$ 

10. Consider the following subspaces of $(\mathbb{F}_3)^3$:

$$U = \text{Span}\{(1, -2, 2), (1, -1, 0)\}; \quad W = \text{Span}\{(1, 2, 0)\}.$$ 

(a) Verify that $(\mathbb{F}_3)^3 = U \oplus W.$

(b) Observe that $\mathcal{B} = \{(1, -2, 2), (1, -1, 0), (1, 2, 0)\}$ is a basis of $V$ given by juxtaposing a basis of $U$ and a basis of $W$. Give the change of basis matrix $\mathcal{B}M_{\mathcal{B}}$.

(c) Compute the coordinate matrices $[P_{U,W}]_{\mathcal{B}}$ and $[P_{U,W}]_{\mathcal{S}t}$ for the projection $P_{U,W}$ relative to the bases $\mathcal{B}$ and $\mathcal{S}t$, respectively. How are these two matrices related?

11. Let $\phi : V \to V$ be a linear operator such that $\phi^2 = \phi$. Prove that $V = \text{im} \phi \oplus \ker \phi$, and that $\phi = P_{\text{im} \phi, \ker \phi}$. That is, $\phi$ is a projection.

12. The number of $2 \times 2$ matrices with coefficients in the finite field $\mathbb{F}_p$ is $p^4$. What fraction of them are invertible? (You must justify your answer.) Deduce that as $p \to \infty$, the probability that a randomly chosen $2 \times 2$ matrix is invertible tends to 1.

13. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in \mathbb{R}$. Prove that $\phi$ is a linear map. (Here $\mathbb{R}$ is viewed as a 1-dimensional real vector space.) Is the statement still true if we relax the continuity hypothesis?

14. If $V$ is a vector space over the field $F$, a **linear functional** is a linear map $\phi : V \to F$. We define the **dual space** $V^* := \mathcal{L}(V, F)$ to be the vector space of all linear functionals $\phi : V \to F$ with the usual addition and scalar multiplication of linear maps. I will give you a few basic facts regarding the dual space and then ask you to prove something.

If $V$ is finite dimensional and $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of $V$, there exists a unique basis $\mathcal{B}^* = \{v_1^*, \ldots, v_n^*\}$ of $V^*$ such that $v_i^*(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

We call $\mathcal{B}^*$ the **dual basis** for $\mathcal{B}$. (Since a linear map is uniquely determined by its values on a basis, the above equation uniquely defines $v_i^* : V \to F$.)

If $\phi : V \to W$ is a linear map, there exists a unique linear map $\phi^* : W^* \to V^*$ such that $\phi^*(f)(v) = f(\phi(v))$ for all $f \in W^*$ and $v \in V$.

We call $\phi^*$ the **dual** of $\phi$. (You should check for yourself that $\phi^*$ is a linear map.)

Now assume $V, W$ are both finite dimensional, and let $\mathcal{B}$ and $\mathcal{C}$ be bases of $V$ and $W$, respectively. If $[\phi]_{\mathcal{B}}$ is the coordinate matrix of $\phi$, prove that the coordinate matrix of the dual map satisfies $\mathcal{B}^*[\phi^*]_{\mathcal{C}}^* = (\phi[\phi]_{\mathcal{B}})^T$, where $A^T$ denotes the **transpose** of the matrix $A$.

15. Suppose $\phi : V \to W$ is a linear map between arbitrary vector spaces, and let $\phi^* : W^* \to V^*$ be the dual map. (See the previous exercise for the definition.)

(a) Prove that $\phi$ is injective if and only if $\phi^*$ is surjective.

(b) Prove that $\phi$ is surjective if and only if $\phi^*$ is injective.