MATH 236 / WINTER 2009
ASSIGNMENT 8: DUE MONDAY APRIL 6, 2009

Please write your solutions neatly and bind them with a staple. Unless otherwise stated, you must carefully justify everything you write. I encourage you to work in groups on these, but you must write your solutions in your own words. The exercise(s) marked with a ⋆ are optional, and they will count as a bonus if you solve them correctly. Your solutions are due at the end of class. No late assignments will be accepted.

Unless otherwise specified all vector spaces are finite dimensional over a field \( F \).

1. Let \( V \) be a finite dimensional vector space over an algebraically closed field \( F \). Suppose \( \phi : V \to V \) is a linear operator with eigenvalues \( \lambda_1, \ldots, \lambda_r \) of multiplicity \( m_1, \ldots, m_r \), respectively. Prove that \( \det(\phi) = \lambda_1^{m_1} \cdots \lambda_r^{m_r} \). (Colloquially, we say the determinant of \( \phi \) is the product of its eigenvalues.)

2. Suppose \( V \) is an inner product space and \( v, v' \in V \). Prove that \( v = v' \) if and only if \( \langle v, w \rangle = \langle v', w \rangle \) for all \( w \in V \).

3. Suppose \( V \) is a real inner product space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). Prove that for all vectors \( v, w \in V \), we have
   \[
   \langle v, w \rangle = \frac{1}{4} \| v + w \|^2 - \frac{1}{4} \| v - w \|^2.
   \]
   This is called the **polarization identity**. It allows one to recover the inner product by knowing only the associated norm. (There is a more complicated formula for complex inner product spaces.)

4. Suppose \( V \) is an inner product space and \( \{v_1, \ldots, v_n\} \) is an orthogonal list of vectors. Prove that
   \[
   \|v_1 + \cdots + v_n\|^2 = \|v_1\|^2 + \cdots + \|v_n\|^2.
   \]
   This is called the **Pythagorean Theorem** for inner product spaces. (Draw a picture with \( V = \mathbb{R}^2 \) and \( n = 2 \) to see why.)

5. Let \( W \) be the subspace of \( \mathbb{R}^2 \) spanned by the vector \((3, 4)\). Using the standard inner product, let \( \phi \) be the orthogonal projection of \( \mathbb{R}^2 \) onto \( W \). Find
   (a) a formula for \( \phi(x_1, x_2) \);
   (b) the matrix of \( \phi \) in the standard basis;
   (c) \( W^\perp \)
   (d) an orthonormal basis in which \( \phi \) is represented by the matrix \[
   \begin{bmatrix}
   1 & 0 \\
   0 & 0
   \end{bmatrix}.
   \]

6. Describe explicitly all inner products on \( V = \mathbb{C}^1 \).
7. Let $V = \mathbb{C}^2$. For each matrix $Q$ below, determine if the pairing

$$\langle \alpha, \beta \rangle = \beta^* Q \alpha = \sum_{i,j=1}^{2} \alpha_i Q_{ij} \beta_j.$$ is an inner product. Here $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^2$ and similarly for $\beta$.

(a) $Q = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}$
(b) $Q = \begin{bmatrix} i & i \\ -i & 2 \end{bmatrix}$
(c) $Q = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

8. One can use the Cauchy/Schwarz inequality to deduce other geometric inequalities that are not immediately obvious. As an example, if $a,b,c$ are positive real numbers, then

$$\frac{a+b+c}{\sqrt{3}} \leq \sqrt{a^2 + b^2 + c^2}.$$ (†)

This means the sum of the distinct side lengths of a rectangular box divided by $\sqrt{3}$ is at most the length of a diagonal of the box. (Draw a picture of the box with one vertex at the origin in $\mathbb{R}^3$ to see it.)

(a) Using the Cauchy/Schwartz inequality (or otherwise), show that for any real numbers $a_1, a_2, \ldots, a_N$, we have

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n \leq \left( \sum_{n=1}^{N} a_n^2 \right)^{1/2}.$$

(b) Deduce the inequality (†).

9. Let $V = \mathbb{R}[X]_3$ be the subspace of $\mathbb{R}[X]$ of polynomials of degree at most 3. Equiv $V$ with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t) \, dt.$$ (a) Find the orthogonal complement of the subspace of constant polynomials.

(b) Apply the Gram/Schmidt process to the basis $\{1, x, x^2, x^3\}$.

10. Let $V$ be the real inner product space consisting of the space of real-valued continuous functions on the interval $-1 \leq t \leq 1$, with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t) \, dt.$$ Let $W$ be the subspace of odd functions. Find the orthogonal complement of $W$. (Explain your answer carefully.)