THE POINCARÉ-LELONG FORMULA ON ALGEBRAIC CURVES

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Abstract. The Poincaré-Lelong formula on the Berkovich analytic space associated to a smooth proper curve $X$ over a complete algebraically closed nonarchimedean field $K$ relates the Laplacian of $\varphi$ to the divisor of the rational function $\varphi$. The Laplacian depends on a choice of metric associated to the Berkovich space. Here we show that if $X \cong \mathbb{P}^1$ or $K$ has countable residue field, the canonical metric is the unique metric for which the Poincaré-Lelong formula can hold. This is a rapidly written proof developed from discussions at the Arizona Winter School. It is not intended to be a complete account of the problem; it is an account of how we executed a sketch of the proof indicated by Matthew Baker.

1. THE POINCARÉ-LELONG HYPOTHESIS

Let $K$ be an algebraically closed field complete with respect to a nontrivial nonarchimedean valuation. Suppose $X$ is a connected smooth proper curve over $K$.

The hyperbolic space associated to $X_{\text{Berk}}$, denoted $H(X_{\text{Berk}})$, is endowed with a canonical metric $\rho$ coming from the invariance of the modulus of Berkovich annuli. Note that $\rho$ does not give the subspace topology on $H(X_{\text{Berk}})$ induced from $X_{\text{Berk}}$. The topology on $X_{\text{Berk}}$ is induced by an arboreal system $\{\Gamma_\alpha\}$, and $\rho$ metrizes each of the compact locally finite topological graphs $\Gamma_\alpha$.

For any metric $\theta$ on $H(X_{\text{Berk}})$, we have the classes of functions $\text{BDV}(X_{\text{Berk}}, \theta)$ and $\text{CPA}(X_{\text{Berk}}, \theta)$, denoting the functions of bounded differential variation and the continuous piecewise affine functions, respectively. Both of these concepts depend on the choice of metric $\theta$. In order to display the dependence of the Laplacian $\Delta$ on the metric, we will write $\Delta_\theta$ unless it is clear from context which metric we are using.

The result that motivates the present discussion comes from Thuillier’s thesis:

**Theorem 1** ([4, Prop. 3.3.15]). For any rational function $\varphi \in K(X)$, the function $z \mapsto -\log_v |\varphi(z)|$ is an element of $\text{BDV}(X_{\text{Berk}}, \rho)$, and

$$\Delta(-\log_v |\varphi|) = \delta_{\text{div}(\varphi)}.$$  \hfill (1)

The equation (1) is called the Poincaré-Lelong formula because of an analogous formula from complex analysis. We wish to investigate the dependence of the Poincaré-Lelong formula on the metric on $H(X_{\text{Berk}})$. As the right side of (1) is independent of the metric, while the left depends heavily on it, one might conjecture that the canonical metric is the only one for which the Poincaré-Lelong formula could hold. Let us make this precise by defining the following Poincaré-Lelong hypothesis on a metric $\theta$ on $H(X_{\text{Berk}})$:
Thuillier’s theorem can be simply state by saying that the canonical metric $\rho$ satisfies the Poincaré-Lelong hypothesis (PL).

With these definitions, we can state our

**Metric Rigidity Theorem.** If either $X \cong \mathbb{P}^1$ or the residue field of $K$ is countable, then the canonical metric is the unique metric on $X_{\text{Berk}}$ satisfying the Poincaré-Lelong hypothesis.

The reason for the restriction on $K$ in the theorem is that the method of proof depends on the ability to construct rational functions with specified zeros and poles. This can be done explicitly if $X = \mathbb{P}^1$, but if $X$ has higher genus, then we are restricted by our use of the following theorem of Rumely:

**Theorem 2** ([3, Thm. 1.3.1]). Suppose $K$ has countable residue field, and let $X$ be a connected smooth proper curve over $K$. Let $U \subset X_{\text{Berk}}$ be an open neighborhood and let $\zeta \in X(K) \setminus U(K)$ be any classical point. Then there exists a non-constant rational function $\varphi \in K(X)$ such that all of the zeros of $\varphi$ lie in $U$, and the only pole of $\varphi$ is at $\zeta$.

To prove the Metric Rigidity Theorem, we proceed first by showing that the metric must scale by a constant on segments of graphs $\Gamma$ in the arboreal system. Then we compute the Laplacian on large graphs in the arboreal system in order to show that the scaling constant must be 1. In each of these steps, we need to choose a rational function $\varphi$ that is non-constant on a particular segment. Throughout the remainder of the proof, we assume that $\theta$ is a metric on $H(X_{\text{Berk}})$ satisfying the Poincaré-Lelong hypothesis.

By way of notation, if $e$ is an oriented segment of a graph $\Gamma$ in the arboreal system of $X_{\text{Berk}}$ (with respect to some unspecified vertex set), then we have two parametrizations $\sigma : e \to [0, \ell]$ and $\sigma_1 : e \to [0, \ell_1]$, with respect to the metrics given by $\rho$ and $\theta$, respectively. We define the reparametrization map $\tau : [0, \ell] \to [0, \ell_1]$ by $\tau = \sigma_1 \circ \sigma^{-1}$. Also, in order to ease notation, we will identify $e$ with $i_\Gamma(e)$, its image under the canonical inclusion $i_\Gamma : \Gamma \to H(X_{\text{Berk}})$.

**Lemma.** Let $\Gamma$ be an element of the arboreal system of $X_{\text{Berk}}$, viewed as a compact locally finite topological tree. Suppose $f : X_{\text{Berk}} \to \mathbb{R} \cup \{\pm \infty\}$ is a non-constant function such that $f$ is real-valued off of $X(K)$, and such that on some segment $e$ of $\Gamma$, $f$ is affine for both $\rho$ and $\theta$. Then the reparametrization map $\tau$ is linear.

**Proof.** Write $f \circ \sigma^{-1}(s) = as + b$ and $f \circ (\sigma_1)^{-1}(s) = a_1 s + b_1$. As $f$ is non-constant, $a_1 \neq 0$, and

$$as + b = (f \circ (\sigma_1)^{-1}) \circ (\sigma_1 \circ \sigma^{-1})(s) = a_1 \tau(s) + b_1.$$

Setting $s = 0$ shows $b = b_1$. Then $as = a_1 \tau(s)$, so that $\tau(s) = \frac{a}{a_1}s$. \qed

For $\Gamma$ a graph in the arboreal system of $X_{\text{Berk}}$, let $r_\Gamma : X_{\text{Berk}} \to \Gamma$ be the canonical retraction map. Also, write $-\log_{\rho} |\varphi|$ for the restriction of $-\log_{\rho} |\varphi|$ to the graph $\Gamma$, identified in the obvious way with its image under $i_\Gamma$.

**Proposition 1.** Suppose that either $X \cong \mathbb{P}^1$ or that $K$ has countable residue field. Let $\Gamma$ be any graph in the arboreal system of $X_{\text{Berk}}$, and take any oriented segment

(PL) For any $\varphi \in K(X)$, the function $-\log_{\rho} |\varphi|$ lies in BDV$(X_{\text{Berk}}, \theta)$, and

$$\Delta_\theta(-\log_{\rho} |\varphi|) = \delta_{\text{div}(\varphi)}.$$
e of $\Gamma$ with type II endpoints $x_1$ and $x_2$. Then there exists $\varphi \in K(X)$ such that the map $z \mapsto -\log_v |\varphi(z)|$ is affine with nonzero slope on $e$ with respect to both $\rho$ and $\theta$. Moreover, it can be chosen so that $(r_1)_*\Delta(-\log_v |\varphi|) = N\delta_{x_1} - N\delta_{x_2}$ for some $N > 0$, the Laplacian being defined with respect to either $\rho$ or $\theta$.

Proof. We treat the case $X \cong \mathbb{P}^1$ first. Here $e$ is the minimal connected closed subset containing both $x_i$. Let $\Lambda$ be the image of the unique minimal path connecting two classical points $z_1$ and $z_2$, where we choose $z_1$ and $z_2$ so that $r_1(z_i) = x_i$. If we choose coordinates on $X$ so that $z_1$ corresponds to $0$ and so that $z_2$ corresponds to $\infty$, then $\varphi(z) = z$ will have the desired properties. Indeed, we know that $-\log_v |\varphi|$ is affine with slope $-1$ along $e$ for the canonical metric (by direct computation), and $\Delta_\rho(-\log_v |\varphi|) = \delta_{z_1} - \delta_{z_2}$ by Thuillier’s theorem. But $\theta$ also satisfies the Poincaré-LeLong hypothesis, so $\Delta_\theta(-\log_v |\varphi|) = \delta_{z_1} - \delta_{z_2}$. We conclude that $(r_1)_*\Delta(-\log_v |\varphi|) = \delta_{x_1} - \delta_{x_2}$ for both metrics. As $(r_1)_*\Delta(-\log_v |\varphi|) = \Delta(-\log_v |\varphi|_1)$, we conclude that $-\log_v |\varphi|$ is continuous piecewise affine on $\Gamma$ with respect to both metrics. This follows from the fact that if the Laplacian of a function on a metrized graph is discrete, then the function is continuous piecewise affine (see [2, Cor. 5.3]). Finally, observe that $-\log_v |\varphi|$ must have constant slope on $e$ since otherwise it’s Laplacian would have an extra Dirac mass somewhere on the interior of $e$.

Now consider the case where $K$ has countable residue field and $X$ is arbitrary. For $i = 1, 2$, select open sets $U_i$ of $X_{\text{Berk}}$ each isomorphic to $\mathcal{B}(0,1)^-$ such that $x_i$ is the unique boundary point of $U_i$ and such that the corresponding tangent direction doesn’t lie in $T_{\Gamma,x_i}$. Such open sets exist by the fact that each $x_i$ is type II. (Consider a neighborhood of $x_i$ isomorphic either to an open Berkovich disk or to an annulus.) The canonical retraction map $r_1 : X_{\text{Berk}} \to \Gamma$ collapses $U_i$ to $x_i$.

By Theorem 2, we can find $\varphi \in K(X)$ such that the zeros of $\varphi$ all lie in $U_1$ and such that $\varphi$ has a unique pole in $U_2$. As $\rho$ and $\theta$ both satisfy (PL), we have

\[(r_1)_*\Delta(-\log_v |\varphi|_1) = (r_1)_*\delta_{\text{div}(\varphi)} = N\delta_{x_1} - N\delta_{x_2},\]

where $N = \deg(\text{div}(\varphi)_0)$. By coherence of measures on the arboreal system, this must be equal to $\Delta(-\log_v |\varphi|_1)$. The argument in the case $X \cong \mathbb{P}^1$ now shows that $-\log_v |\varphi|$ is affine on $e$ for both metrics. The maximum modulus principle implies that $-\log_v |\varphi|_1$ attains its maximum value at $x_1$ and its minimum value at $x_2$. If $-\log_v |\varphi|$ were constant on $e$, then it would be constant on $\Gamma$. But the Laplacian of a constant function is the zero measure, contradicting (2).

\[\square\]

**Proposition 2.** Suppose that either $X \cong \mathbb{P}^1$ or that $K$ has countable residue field. Let $\Gamma$ be a graph in the arboreal system of $X_{\text{Berk}}$ and suppose $e$ is an oriented segment of $\Gamma$ with type II endpoints. Then the reparametrization map $\tau$ has slope 1.

\[\text{Proof.}\] By Proposition 1, there exists a rational function $\varphi$ so that the function $-\log_v |\varphi|$ is affine with nonzero slope on $e$ for both $\rho$ and $\theta$. Lemma implies that the corresponding reparametrization map is linear, say $\tau(s) = cs$. We must show that $c = 1$. Note that $c$ is independent of $\varphi$.

Choose $\rho \in e$ to be a type II point of $X_{\text{Berk}}$, and suppose further that it is not one of the endpoints of $e$. Let $q$ be the terminal point of $e$. We may apply Proposition 1 again to find a new rational function $\varphi$ such that $(r_0)_*\Delta(-\log_v |\varphi|) = N\delta_q - N\delta_p$ for some $N > 0$. Set $f = -\log_v |\varphi| \circ \sigma^{-1}$ and set $f_1 = -\log_v |\varphi| \circ \sigma_1^{-1}$. Let $\nu_1$ and
\[ \vec{v}_1, \vec{v}_2 \] be the two tangent directions to \( \Gamma \) at \( p \). Then
\[
N = \Delta_{\rho}(-\log_{\varepsilon}|\varphi|)(\sigma(p)) \quad \text{since } \rho \text{ satisfies (PL)},
\]
\[
= d_{\vec{v}_1}f(\sigma(p)) + d_{\vec{v}_2}f(p)
\]
\[
= d_{\vec{v}_1}(f_1 \circ \tau)(\sigma(p)) + d_{\vec{v}_2}(f_1 \circ \tau)(\sigma(p))
\]
\[
= \frac{d\tau}{ds}(\sigma(p)) \left( d_{\vec{v}_1}f_1(\sigma_1(p)) + d_{\vec{v}_2}f_1(\sigma_1(p)) \right)
\]
\[
= c\Delta_{\theta}(-\log_{\varepsilon}|\varphi|)(p)
\]
\[
= cN \quad \text{since } \theta \text{ satisfies (PL)}.
\]

Note that in the second, third and fourth lines we are calculating the Laplacian on a metrized graph with metrics implicitly given by the parametrizations \( \sigma \) and \( \sigma_1 \). As \( N \neq 0 \), we are finished.

**Proof of the Metric Rigidity Theorem.** Choose any graph \( \Gamma \) in the arboreal system for \( X_{\text{Berk}} \). By the previous proposition, we know that \( \theta \) and \( \rho \) agree on any segment of \( \Gamma \) with type II endpoints. As type II points are dense in the \( \rho \)-metric topology, we can always find a vertex set for \( \Gamma \) consisting consisting entirely of type II points, perhaps after enlarging \( \Gamma \) slightly if it has type III or IV points of valence 1. Thus \( \rho \) and \( \theta \) agree on a cofinal subsystem of the arboreal system for \( X_{\text{Berk}} \). A metric on \( \mathbf{H}(X_{\text{Berk}}) \) is determined by its restriction to any graph in the arboreal system, so the proof is complete.

\[ \square \]

2. **Final remarks**

One could remove the hypothesis that \( K \) has countable residue field by reprove Proposition 1 without the use of Rumely’s theorem, as this is the only place where the hypothesis was explicitly used. One alternative strategy suggested by Matt is to use his result from [1] giving surjectivity of principal Cartier divisors on a curve \( X \) onto the group of principal divisors of its reduction graph for some fixed model \( \mathfrak{X} \). This requires one to assume that \( X \) is defined over a discretely valued subfield of \( K \) (with the restriction of the valuation being nontrivial). Any curve \( X \) over \( K \) is certainly defined over a field \( F \subset K \) finitely generated over the prime field. Is \( F \) discretely valued, at least if one throws in some \( \alpha \in K \) such that \( |\alpha| < 1 \) (in case \( |F^\times| = 1 \)?)

**References**


