UNIFORM BOUNDS ON PRE-IMAGES UNDER QUADRATIC DYNAMICAL SYSTEMS

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Abstract. For any elements \(a, c\) of a number field \(K\), let \(\Gamma(a, c)\) denote the backwards orbit of \(a\) under the map \(f_c: \mathbb{C} \to \mathbb{C}\) given by \(f_c(x) = x^2 + c\). We prove an upper bound on the number of elements of \(\Gamma(a, c)\) whose degree over \(K\) is at most some constant \(B\). This bound depends only on \(a\), \([K: \mathbb{Q}]\), and \(B\), and is valid for all \(a\) outside an explicit finite set. We also show that, for any \(N \geq 4\) and any \(a \in K\) outside a finite set, there are only finitely many pairs \((y_0, c) \in \mathbb{C}^2\) for which \([K(y_0, c): K] < 2^{N-3}\) and the value of the \(N\)th iterate of \(f_c(x)\) at \(x = y_0\) is \(a\). Moreover, the bound \(2^{N-3}\) in this result is optimal.

1. Introduction

1.1. Bounding the Number of Pre-Images. For an elliptic curve \(E\) over a number field \(K\), the Mordell–Weil theorem implies finiteness of the group \(E_{\text{tors}}(K)\) of \(K\)-rational torsion points on \(E\). Merel [8], building on work of Mazur, Kamienny, and others, proved that \(#E_{\text{tors}}(K)\) is bounded by a function of \([K: \mathbb{Q}]\) (uniformly over all \(K\) and \(E\)). This implies the following uniform bound on torsion points over extensions of \(K\) of bounded degree (see [10, Cor. 6.64]):

**Theorem 1.1.** Fix positive integers \(B\) and \(D\). There is an integer \(\lambda(B, D)\) such that for any number field \(K\) with \([K: \mathbb{Q}] \leq D\), and for any elliptic curve \(E/K\), we have

\[ \# \{ P \in E(K) : [K(P) : K] \leq B \text{ and } \exists N \geq 1 \text{ s.t. } [N]P = \mathcal{O} \} \leq \lambda(B, D). \]

From a dynamical perspective, Theorem 1.1 controls the number of bounded-degree pre-images of the point \(\mathcal{O}\) under the various maps \([N]: E \to E\). In this paper we prove an analogue of this result for maps \(\mathbb{A}^1 \to \mathbb{A}^1\) defined by the iterates of a degree-2 polynomial \(f \in \mathbb{Q}[x]\). Write \(f^N\) for the \(N\)th iterate of the polynomial \(f\). A height argument similar to the one used by Mordell and Weil shows that, for any number field \(K\), any quadratic \(f \in K[x]\), and any \(a \in K\) and \(B > 0\), the set

\[ \{ x_0 \in \overline{K} : [K(x_0) : K] \leq B \text{ and } f^N(x_0) = a \text{ for some } N \geq 1 \} \]

is finite. The sizes of these sets cannot be bounded in terms of \(K\), \(a\), and \(B\): for any \(N \geq 1\), put \(f(x) := (x - b)^2 + b\) where \(b := a - 2^{2N}\), and note that \(f^N(b + 2) = a\). However, we will prove such a bound on these sets in case \(f\) varies over the family of polynomials

\[ f_c(x) := x^2 + c. \]

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Theorem 1.2. Fix positive integers $B$ and $D$. For all but finitely many values $a \in \mathbb{Q}$, there is an integer $\kappa(B, D, a)$ with the following property: for any number field $K$ such that $[K: \mathbb{Q}] \leq D$ and $a \in K$, and for any $c \in K$, we have
\[
\# \{x_0 \in \overline{\mathbb{Q}} : [K(x_0) : K] \leq B \text{ and } f_c^N(x_0) = a \text{ for some } N \geq 1 \} \leq \kappa(B, D, a).
\]

Further, we give an explicit description of the excluded values $a$: they are the critical values of the polynomials $f_c^j(0) \in \mathbb{Z}[x]$, for $2 \leq j \leq 4 + \log_2(BD)$. It follows that the number of such values is less than $16BD$, and we will show that these values do not have the form $\alpha/m$ with $\alpha$ an algebraic integer and $m$ an odd integer. We do not know whether the result would remain true if we did not exclude these finitely many values $a$. We prove that this is the case if $B = D = 1$ (see Theorem 4.1).

We do not assert any uniformity in $a$ in Theorem 1.2, and in fact such uniformity cannot hold (since $a$ can be chosen as $f_c^N(x_0)$ for fixed $c, N, x_0$). Also, our proof gives no explicit bound on the constant $\kappa(B, D, a)$, since we use a noneffective result due to Vojta (which generalizes the Mordell conjecture). Our proof of Theorem 1.2 carries over immediately to the family of polynomials $g_c(x) := x^k + c$ for any fixed $k \geq 2$; it would be interesting to analyze other families of polynomials.

In a different direction, if we fix $N$ and vary $c$, the choices of $B$ and $D$ become crucial:

Theorem 1.3. Let $K$ be a number field and fix $a \in K$ and $N \geq 4$. There is a finite extension $L$ of $K$ for which infinitely many pairs $(y_0, c) \in \overline{K} \times K$ satisfy $f_c^N(y_0) = a$ and $[L(y_0, c) : L] \leq 2^{N-3}$. Conversely, if $a$ is not a critical value of $f_c^j(0)$ for any $2 \leq j \leq N$, then only finitely many pairs $(y_0, c) \in \overline{K} \times K$ satisfy $f_c^N(y_0) = a$ and $[K(y_0, c) : K] < 2^{N-3}$.

In this result, some values $a$ must be excluded: for $a = -1/4$, we will show that infinitely many pairs $(y_0, c) \in \overline{\mathbb{Q}} \times \overline{\mathbb{Q}}$ satisfy $f_c^N(y_0) = a$ and $[\mathbb{Q}(y_0, c) : \mathbb{Q}] \leq 2^{N-4}$. Note that $a = -1/4$ is the unique critical value of $f_c^2(0) = c^2 + c = (c+1/2)^2 - 1/4$. If we fix $c$ (and $N$ and $a$), then only finitely many $y_0 \in \overline{\mathbb{Q}}$ satisfy $f_c^N(y_0) = a$; thus the first part of Theorem 1.3 would remain true if we required the occurring values of $c$ to be distinct. We will discuss Theorem 1.3 further in the next subsection after defining the analogues of modular curves for this problem.

A different dynamical analogue of Merel’s result has been conjectured by Morton and Silverman [9]. For a field $K$ and a non-constant endomorphism $\phi$ of a variety $V$ over $K$, define the set of preperiodic points for $\phi$ to be
\[
\text{PrePer}(\phi) = \{ P \in V(K) : \phi^N(P) = \phi^M(P) \text{ for some } N > M \geq 0 \}.
\]

In case $V$ is an elliptic curve and $\phi = [R]$ for some $R > 1$, the set $\text{PrePer}(\phi)$ coincides with $V_{\text{tors}}(K)$. This motivates the following special case of the Morton–Silverman conjecture:

Conjecture. For any positive integer $D$, there is an integer $\mu(D)$ such that, for all number fields $K$ of degree at most $D$ and all $c \in K$, we have
\[
\#(\text{PrePer}(f_c) \cap \mathbb{A}^1(K)) \leq \mu(D).
\]

See [10, §3.3] for a discussion of this conjecture.

1.2. Notation, Pre-Image Curves, and the Proof Strategy. Let $K$ be a field whose characteristic is not 2. For $c \in K$, view $f_c(x) := x^2 + c$ as a mapping
\[ \mathbb{A}_K^1 \to \mathbb{A}_K^1 \]. We will study the dynamics of this mapping, by which we mean the behavior of points under repeated application of this map.

In order to prove results valid for all \( c \in K \), it is convenient to first treat \( c \) as an indeterminate. This will be our convention unless otherwise specified.

**Definition 1.4.** Fix an element \( a \in K \) and a positive integer \( N \). We write \( Y^\text{pre} (N,a) \) for the algebraic set in \( \mathbb{A}^2 \) defined by \( f_N^2(x) = a \). If \( Y^\text{pre} (N,a) \) is geometrically irreducible (that is, irreducible over \( \overline{K} \)), we define the \( N \)th **pre-image curve** \( X^\text{pre} (N,a) \) to be the completion of the normalization of \( Y^\text{pre} (N,a) \).

Note that a point \( (x_0,a_0) \in \mathbb{A}^2(\overline{K}) \) lies on \( Y^\text{pre} (N,a) \) if and only if \( x_0 \) is a pre-image of \( a \) under the \( N \)th iterate of the map \( x \mapsto f_{a_0}(x) \). For example, since the map \( x \mapsto f_{a-a^2}(x) \) fixes \( x = a \), the point \((a, a-a^2)\) lies on \( Y^\text{pre} (N,a) \) for every \( N \geq 1 \). Likewise, since \( f_{-a^2-a-1} \) maps

\[ a \mapsto -a - 1 \mapsto a, \]

for every \( N \geq 1 \) the points \((a, a-a^2 - a - 1)\) and \((a - 1, -a^2 - a - 1)\) lie on \( Y^\text{pre} (2N,a) \) and \( Y^\text{pre} (2N-1,a) \), respectively.

The following result gives a sufficient condition for irreducibility of \( Y^\text{pre} (N,a) \).

**Theorem 1.5.** Suppose \( N \) is a positive integer and \( a \in K \) is not a critical value of \( f_j^2(0) \) for any \( 2 \leq j \leq N \). Then \( Y^\text{pre} (N,a) \) is geometrically irreducible, and the genus of \( X^\text{pre} (N,a) \) is \( (N-3)2^{N-2} + 1 \).

We now restate the main part of Theorem 1.3:

**Corollary 1.6.** Let \( K \) be a number field and fix \( N \geq 4 \) and \( a \in K \) that is not a critical value of \( f_j^2(0) \) for any \( 2 \leq j \leq N \). Then only finitely many \( P \in X^\text{pre} (N,a)(\overline{K}) \) satisfy \( [K(P) : K] < 2^{N-3} \), but there is a finite extension \( L \) of \( K \) for which infinitely many \( P \in X^\text{pre} (N,a)(\overline{K}) \) satisfy \( [L(P) : L] = 2^{N-3} \).

This result should be compared with a conjecture of Abramovich and Harris [1, p. 229], which says that a curve \( C \) over a number field \( K \) admits a rational map of degree at most \( d \) to a curve of genus 0 or 1 if and only if there is a finite extension \( L \) of \( K \) for which infinitely many \( P \in C(\overline{L}) \) satisfy \( [L(P) : L] = d \). In light of the above result, this conjecture says that \( 2^{N-3} \) should be the minimal degree of any rational map from \( X^\text{pre} (N,a) \) to a curve of genus 0 or 1. We will prove that this is in fact the case (one minimal degree map is the composition \( \delta_1 \circ \delta_2 \circ \cdots \circ \delta_N \), whose image is the genus 1 curve \( X^\text{pre} (3,a) \), where the maps \( \delta_M \) are defined below). It should be noted, however, that Debarre and Fahlaoui have produced counterexamples to the Abramovich–Harris conjecture [3, 5,17]. Still, the conjecture is known to be true when \( d \) is small (due to Abramovich, Harris, Hindry, Silverman, and Vojta), and it is important to understand when it holds.

Define a degree-2 morphism \( \delta : \mathbb{A}^2 \to \mathbb{A}^2 \) by \( \delta(x,c) = (x^2 + c, c) \). For \( N > 1 \), let \( \delta_N \) be the restriction of \( \delta \) to \( Y^\text{pre} (N,a) \), so the image of \( \delta_N \) is \( Y^\text{pre} (N-1,a) \). For any fixed \( a \in K \), this gives a tower of algebraic sets and maps

\[
\cdots \delta_{N+1} Y^\text{pre} (N,a) \xrightarrow{\delta_N} Y^\text{pre} (N-1,a) \xrightarrow{\delta_{N-1}} \cdots \xrightarrow{\delta_2} Y^\text{pre} (1,a).
\]

When \( Y^\text{pre} (N,a) \) and \( Y^\text{pre} (N-1,a) \) are geometrically irreducible, \( \delta_N \) induces a degree-2 morphism \( \delta_N : X^\text{pre} (N,a) \to X^\text{pre} (N-1,a) \).

Our strategy for proving Theorem 1.2 in case \( B = D = 1 \) is as follows: if \( a \in \mathbb{Q} \) is not a critical value of \( f_j^2(0) \) for any \( j \in \{2,3,4\} \), then Theorem 1.5 implies
that $X^{\text{pre}}(4, a)$ is a geometrically irreducible curve of genus 5. By the Mordell conjecture (Faltings’ theorem [4]), $X^{\text{pre}}(4, a)(\mathbb{Q})$ is finite. An argument involving heights shows that any point in $\mathbb{A}^2(\mathbb{Q})$ has a total of finitely many pre-images in $\mathbb{A}^2(\mathbb{Q})$ under the various iterates of $\delta$. Thus the union of all $Y^{\text{pre}}(N, a) \mathbb{Q})$ with $N \geq 4$ is finite. To deduce Theorem 1.2 in case $B = D = 1$, note that for each $N < 4$ the number of points in $Y^{\text{pre}}(N, a)(\mathbb{Q})$ having fixed values of $a$ and $c$ is at most $2^N$, and in particular is bounded independently of $c$. The proof of Theorem 1.2 for other values of $B$ and $D$ follows the same strategy, but instead of Faltings’ theorem we use a consequence of Vojta’s inequality on arithmetic discriminants [14]; this requires some additional arguments adapting Vojta’s result to our situation.

We remark that the algebraic sets $Y^{\text{pre}}(N, 0)$ have arisen previously in the context of the $p$-adic Mandelbrot set [6]. Also the sets $Y^{\text{pre}}(2, a)$ occur implicitly in the study of uniform lower bounds on canonical heights of morphisms [5]; we will discuss the connection between such bounds and our results in Remark 4.9.

The remainder of the paper is organized as follows. In §2 we give a criterion for nonsingularity of $Y^{\text{pre}}(N, a)$ and prove that nonsingularity implies irreducibility. In §3, in case $Y^{\text{pre}}(N, a)$ is nonsingular, we compute the genus of $X^{\text{pre}}(N, a)$, as well as the minimal degree of any rational map from $X^{\text{pre}}(N, a)$ to a curve of genus 0 or 1. We then prove our arithmetic results in §4.

### 2. Smoothness and Irreducibility

In this section we determine when $Y^{\text{pre}}(N, a)$ is nonsingular, and we show that $Y^{\text{pre}}(N, a)$ is irreducible whenever it is nonsingular. Throughout this section, $K$ is an algebraically closed field whose characteristic is not 2.

**Proposition 2.1.** Fix a positive integer $N$. For $a \in K$, the following assertions are equivalent:

(a) $Y^{\text{pre}}(N, a)$ is nonsingular.

(b) $Y^{\text{pre}}(M, a)$ is nonsingular for $1 \leq M \leq N$.

(c) There do not exist an integer $j$ with $2 \leq j \leq N$ and an element $c_0 \in K$ such that

$$f_j^{c_0}(0) = a \quad \text{and} \quad \frac{\partial f_j^{c_0}(0)}{\partial c} \bigg|_{c=c_0} = 0.$$  

**Remark 2.2.** Condition (c) says that $a$ is not a critical value of $f_j^c(0)$ for any $2 \leq j \leq N$.

**Proof.** It suffices to show that (a) and (c) are equivalent, since if (c) holds for some $N$ then it automatically holds for every smaller $N$. In order to prove equivalence of (a) and (c), we must describe the singular points on $Y^{\text{pre}}(N, a)$. A point $(x_0, c_0) \in \mathbb{A}^2(K)$ is a singular point on $Y^{\text{pre}}(N, a)$ if and only if the following three equations are satisfied:

1. $f_N^{c_0}(x_0) = a$
2. $\frac{\partial f_N^{c_0}(x)}{\partial x} \bigg|_{x=x_0} = 0$
3. $\frac{\partial f_N^{c_0}(x_0)}{\partial c} \bigg|_{c=c_0} = 0$.
By repeatedly applying the chain rule (and using that $f'_c(x) = 2x$), we find
\[
\frac{\partial f^N_{c_0}(x)}{\partial x} \bigg|_{x=x_0} = f'_c \left( f^{-1}_{c_0}(x_0) \right) \cdot f'_c \left( f^{-2}_{c_0}(x_0) \right) \cdots f'_c \left( f_{c_0}(x_0) \right) = 2^N \prod_{i=0}^{N-1} f'_c(x_0).
\]

Thus, equation (2) is equivalent to the existence of an integer $i$ with $0 \leq i \leq N - 1$ such that $f^i_{c_0}(x_0) = 0$. For any such $i$, we have
\[
\left. \frac{\partial f^N_{c_0}(x)}{\partial c} \right|_{c=c_0} = \left. \frac{\partial \left( f^N_{c_0}(f^i_{c_0}(x_0)) \right)}{\partial c} \right|_{c=c_0} = \left. \frac{\partial f^{N-i}_{c_0}(y)}{\partial y} \right|_{y=0} \cdot \left. \frac{\partial f^i_{c_0}(x_0)}{\partial c} \right|_{c=c_0} + \left. \frac{\partial f^{N-i}_{c_0}(0)}{\partial c} \right|_{c=c_0}.
\]

Since $f^{N-i}_{c_0}(y) = f^{N-i}_{c_0}(y^2 + c_0)$ is a polynomial in $K[y^2]$, its partial derivative with respect to $y$ has zero constant term, so
\[
\left. \frac{\partial f^N_{c_0}(x_0)}{\partial c} \right|_{c=c_0} = \left. \frac{\partial f^{N-i}_{c_0}(0)}{\partial c} \right|_{c=c_0}.
\]

If $i = N - 1$ then this common value is $\frac{\partial f^{N-i}_{c_0}(0)}{\partial c} = 1$, which in particular is nonzero. Thus, a point $(x_0, c_0) \in \mathbb{K}^2(K)$ is a singular point of $Y_{\text{pre}}(N, a)$ if and only if all three of the following are satisfied:
\begin{enumerate}
  \item[(4)] $f^N_{c_0}(x_0) = a$
  \item[(5)] $f^i_{c_0}(x_0) = 0$ for some $i$ satisfying $0 \leq i \leq N - 2$
  \item[(6)] $\left. \frac{\partial f^{N-i}_{c_0}(0)}{\partial c} \right|_{c=c_0} = 0$.
\end{enumerate}

When (5) holds, equation (4) is equivalent to
\begin{enumerate}
  \item[(7)] $f^{N-i}_{c_0}(0) = a$.
\end{enumerate}

Conversely, if $c_0$ and $i$ satisfy (6) and (7), then there exists $x_0 \in K$ satisfying (5). This implies the equivalence of (a) and (c) (with $j = N - i$).

\begin{remark}
Assertion (c) of Proposition 2.1 gives a criterion for checking whether $Y_{\text{pre}}(N, a)$ is smooth. In fact, it allows us to bound the number of values $a \in K$ for which smoothness fails. Namely, (c) associates to any such value $a \in K$ a pair $(j, c_0)$, where $2 \leq j \leq N$ and $c_0$ is a root of $\frac{\partial f^{N}_{c_0}(0)}{\partial c}$. Since this last polynomial has degree $2^j - 1$, there are at most that many possibilities for $c_0$ corresponding to a specified value $j$. Summing over $2 \leq j \leq N$, we find that $Y_{\text{pre}}(N, a)$ is smooth for all but at most $2^N - N - 1$ values $a \in K$. We checked that equality holds if $K$ has characteristic zero and $N \leq 6$, and we suspect equality holds in most situations. For $2 \leq N \leq 6$, there are precisely $2^N - 1$ nonsingular, and in each case these values $a$ are conjugate over $\overline{\mathbb{Q}}$.
\end{remark}

\begin{corollary}
The algebraic set $Y_{\text{pre}}(1, a)$ is nonsingular for any $a \in K$. The algebraic set $Y_{\text{pre}}(2, a)$ is nonsingular for any $a \in K \setminus \{-1/4\}$.
\end{corollary}

\begin{proposition}
For $a \in K$ and $N \geq 1$, if $Y_{\text{pre}}(N, a)$ is nonsingular then it is irreducible.
\end{proposition}
Proof. First note that $Y_{\text{pre}}(1, a)$ is irreducible for any $a \in K$, since the defining polynomial $x^2 + c - a \in K[x, c]$ is linear in $c$. Henceforth we assume $N > 1$. If $Y_{\text{pre}}(N, a)$ is nonsingular, then Proposition 2.1 implies $Y_{\text{pre}}(M, a)$ is also nonsingular for all $M < N$. We will show that, for $M - 1 < N$, if $Y_{\text{pre}}(M - 1, a)$ is irreducible, then $Y_{\text{pre}}(M, a)$ is irreducible as well. By induction, this implies $Y_{\text{pre}}(N, a)$ is irreducible.

Write the function field of $Y_{\text{pre}}(M - 1, a)$ as $K(y, c)$, where $f^{M-1}_c(y) = a$. The function fields of the components of $Y_{\text{pre}}(M, a)$ are the extensions of $K(y, c)$ defined by the factors of $x^2 + c - y$ in $K(y, c)[x]$. Since each such factor is monic in $x$, and has coefficients in $K[y, c]$, the corresponding component contains a point $(x_0, c_0)$ lying over any prescribed point $(y_0, c_0)$ of $Y_{\text{pre}}(M - 1, a)$. Choose $c_0 \in K$ satisfying $f^{M-1}_{c_0}(c_0) = a$, so $(c_0, c_0)$ is a point of $Y_{\text{pre}}(M - 1, a)$. Then $(0, c_0)$ is the unique point $P \in Y_{\text{pre}}(M, a)$ for which $\delta_M(P) = (c_0, c_0)$. Thus $(0, c_0)$ is contained in each component of $Y_{\text{pre}}(M, a)$, so since $Y_{\text{pre}}(M, a)$ is nonsingular it must be irreducible.

One can also prove this result geometrically: for the key step, note that $\delta_M$ is a finite morphism, so if $Y_{\text{pre}}(M - 1, a)$ is irreducible then $\delta_M$ maps each component of $Y_{\text{pre}}(M, a)$ surjectively onto $Y_{\text{pre}}(M - 1, a)$.

Remark 2.6. In fact, $Y_{\text{pre}}(N, a)$ is typically irreducible even when it is singular. For each $N \geq 1$, the previous two results imply irreducibility of $Y_{\text{pre}}(N, a)$ for all values $a \in K$ not on a short list of potential exceptions. For $N \leq 4$, we checked the values $a$ on these lists, and found that $Y_{\text{pre}}(N, a)$ is irreducible for all $a \in K$ except $a = -1/4$. On the other hand, $Y_{\text{pre}}(N, -1/4)$ has two components for each $N$ with $2 \leq N \leq 6$. We suspect that larger values $N$ behave the same way.

3. Genus and gonality

In this section, for all values of $N$ and $a$ for which $Y_{\text{pre}}(N, a)$ is nonsingular, we compute the genus and gonality of $X_{\text{pre}}(N, a)$. Recall that the gonality is the minimum degree of a non-constant morphism $X_{\text{pre}}(N, a) \to \mathbb{P}^1$. We also compute the minimum degree of a non-constant morphism from $X_{\text{pre}}(N, a)$ to a curve of genus one.

Throughout this section, $K$ is an algebraically closed field whose characteristic is not 2.

For a fixed value $a \in K$, we will compute the genus of $X_{\text{pre}}(N, a)$ inductively, by applying the Riemann-Hurwitz formula to the map $\delta_N: X_{\text{pre}}(N, a) \to X_{\text{pre}}(N - 1, a)$ defined in Section 1. We begin by computing the ramification of this map.

Lemma 3.1. Pick $a \in K$ and $N \geq 2$ for which $Y_{\text{pre}}(N, a)$ is nonsingular. Then $f_c^N(0) = a$ for precisely $2^{N-1}$ values $c \in K$, and the corresponding points $(0, c) \in Y_{\text{pre}}(N, a)(K)$ comprise all points of $X_{\text{pre}}(N, a)(K)$ at which $\delta_N: X_{\text{pre}}(N, a) \to X_{\text{pre}}(N - 1, a)$ ramifies.

Proof. Since $Y_{\text{pre}}(N, a)$ is nonsingular, for each $1 \leq M \leq N$ it follows that $Y_{\text{pre}}(M, a)$ is nonsingular (by Proposition 2.1) and hence irreducible (by Proposition 2.5).

First consider $\delta_N$ on $Y_{\text{pre}}(N, a)$, which is defined by $\delta_N(x, c) = (x^2 + c, c)$. The points with fewer than two pre-images are the images of points with $x = 0$, so
\(\delta_N\) ramifies at precisely the points \((0, c)\) on \(Y^{\text{pre}}(N, a)\). For \(c \in K\), the point \((0, c) \in K^2(K)\) lies on \(Y^{\text{pre}}(N, a)\) if and only if \(f_c^N(0) = a\). Note that \(f_c^N(0) - a\) is a polynomial in \(K[c]\) of degree \(2N - 1\). If \(c_0 \in K\) is a repeated root of \(f_c^N(0) - a\), then

\[
f_{c_0}^N(0) = a \quad \text{and} \quad \frac{\partial f_{c_0}^N(0)}{\partial c} \bigg|_{c=c_0} = 0,
\]

contradicting our nonsingularity hypothesis (by Proposition 2.1). Thus \(f_c^N(0) = a\) for precisely \(2N - 1\) values \(a \in K\), and the corresponding points \((0, c) \in Y^{\text{pre}}(N, a)(K)\) comprise all points of \(Y^{\text{pre}}(N, a)(K)\) at which \(\delta_N\) ramifies.

It remains to show that \(\delta_N\) is unramified at the ‘cusps’ \(X^{\text{pre}}(N, a) \setminus Y^{\text{pre}}(N, a)\). Write the function field of \(X^{\text{pre}}(M, a)\) as \(K(x_M, c)\) where \(x_M^2 + c = x_M - 1\) for \(M > 1\) and \(x_1^2 + c = a\). At the infinite place \(P_1\) of \(K(x_1, c)\), the functions \(x_1\) and \(c\) have poles of orders 1 and 2. Inductively, assume \(x_M\) and \(c\) have poles of orders 1 and 2 at a place \(P\) of \(K(x_M, c)\) which lies over \(P_1\). Then \(y := x_{M+1}/x_M\) satisfies \(y^2 = (x_M - c)/x_M^2\), and since the right side has a nonzero finite value at \(P\), there are two possibilities for the value of \(y\) at \(P\). Thus, Kummer’s theorem [12, Thm. III.3.7] implies that \(P\) lies under two places of \(K(x_{M+1}, c)\), neither of which is ramified. \(\square\)

**Theorem 3.2** (Genus Formula). Let \(a \in K\), and let \(N \geq 1\) be an integer for which \(Y^{\text{pre}}(N, a)\) is nonsingular. Then \(X^{\text{pre}}(N, a)\) is irreducible and has genus \((N - 3)2^{N-2} + 1\).

**Proof.** For each \(M \leq N\), the algebraic set \(Y^{\text{pre}}(M, a)\) is nonsingular (by Proposition 2.1) and hence irreducible (by Proposition 2.5), so also \(X^{\text{pre}}(M, a)\) is irreducible. All that remains is to calculate its genus.

We proceed by induction on \(N\). Let \(g(N)\) denote the genus of \(X^{\text{pre}}(N, a)\). Since \(Y^{\text{pre}}(1, a)\) is defined by \(x^2 + c = a\), it is isomorphic to the \(x\)-line, so \(g(1) = 0\) as desired. Inductively, suppose \(g(N - 1) = (N - 4)2^{N-3} + 1\) for some \(N \geq 2\). We compute \(g(N)\) by applying the Riemann-Hurwitz formula to the degree-2 morphism \(\delta_N: X^{\text{pre}}(N, a) \to X^{\text{pre}}(N - 1, a)\). Lemma 3.1 shows that \(\delta_N\) ramifies at precisely \(2^{N-1}\) points, so

\[
2g(N) - 2 = 2[2g(N - 1) - 2] + \sum_{\text{ramified points of } X^{\text{pre}}(N, a)} 1
\]

\[
= 2[2g(N - 1) - 2] + 2^{N-1},
\]

whence

\[
g(N) = 2g(N - 1) - 1 + 2^{N-2}
\]

\[
= (N - 4)2^{N-2} + 2 - 1 + 2^{N-2}
\]

\[
= (N - 3)2^{N-2} + 1. \quad \square
\]

**Example 3.3.** For a general choice of \(a \in K\), we saw above that \(Y^{\text{pre}}(N, a)\) is irreducible and nonsingular. Passing to the completed curves, the generic picture looks like

\[
\cdots \overset{2-1}{\longrightarrow} X^{\text{pre}}(4, a) \overset{2-1}{\longrightarrow} X^{\text{pre}}(3, a) \overset{2-1}{\longrightarrow} X^{\text{pre}}(2, a) \overset{2-1}{\longrightarrow} X^{\text{pre}}(1, a)
\]

\[
g(4) = 5 \quad g(3) = 1 \quad g(2) = 0 \quad g(1) = 0
\]
The fact that $X^{\text{pre}} (4, a)$ has genus larger than 1 will be of arithmetic value to us in the next section.

For later use, we also summarize the relevant behavior for small values of $N$ and those values of $a$ for which $Y^{\text{pre}} (N, a)$ is singular. We used Magma [2] to compute the data in the following table.

<table>
<thead>
<tr>
<th>$a \in \overline{\mathbb{Q}}$</th>
<th>Algebraic Set</th>
<th>Irreducible Components</th>
<th>Genus</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \in A_2$</td>
<td>$Y^{\text{pre}} (2, -1/4)$</td>
<td>2</td>
<td>0, 0</td>
</tr>
<tr>
<td></td>
<td>$Y^{\text{pre}} (3, -1/4)$</td>
<td>2</td>
<td>0, 0</td>
</tr>
<tr>
<td></td>
<td>$Y^{\text{pre}} (4, -1/4)$</td>
<td>2</td>
<td>1, 1</td>
</tr>
<tr>
<td></td>
<td>$Y^{\text{pre}} (5, -1/4)$</td>
<td>2</td>
<td>5, 5</td>
</tr>
<tr>
<td>$a \in A_3$</td>
<td>$Y^{\text{pre}} (3, a)$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$Y^{\text{pre}} (4, a)$</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$a \in A_4$</td>
<td>$Y^{\text{pre}} (4, a)$</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3.4. We denote by $A_N$ the set of values $a \in \mathbb{Q}$ for which $Y^{\text{pre}} (N, a)$ is singular but $Y^{\text{pre}} (N - 1, a)$ is nonsingular. These sets may be computed using the criterion in Proposition 2.1. For example, $A_2 = \{-1/4\}$. Also $\# A_3 = 3$ and $\# A_4 = 7$. The last column gives the genera of the irreducible components of the given algebraic set.

Remark 3.5. The case $a = -1/4$ is of special interest for various reasons. Here we note that $Y^{\text{pre}} (4, -1/4)$ has infinitely many rational points (since each of its components is the affine part of a rank-one elliptic curve over $\mathbb{Q}$). By contrast, for any other value $a \in \mathbb{Q}$, the above results imply that $Y^{\text{pre}} (4, a)$ is an irreducible curve of genus greater than one, and thus has only finitely many rational points by the Mordell conjecture (Faltings' theorem [4]).

We now compute the gonality of $X^{\text{pre}} (N, a)$:

**Theorem 3.6.** Let $a \in K$, and let $N \geq 2$ be an integer for which $Y^{\text{pre}} (N, a)$ is nonsingular. Then the gonality of $X^{\text{pre}} (N, a)$ is $2^{N-2}$.

Our proof uses Castelnuovo's bound on the genus of a curve on a split surface (see [7, 2.16] or [12, Thm. III.10.3]):

**Theorem 3.7.** Let $C_1$, $C_2$, and $C$ be smooth, projective, geometrically integral curves over $K$, and suppose there is a generically injective map $\psi: C \to C_1 \times_K C_2$. Let $g_i$ be the genus of $C_i$, let $\pi_i$ denote projection from $C_1 \times_K C_2$ onto its $i^{\text{th}}$ factor, and let $n_i$ be the degree of the map $\pi_i \circ \psi: C \to C_i$. Then the genus $g$ of $C$ satisfies

$$g \leq n_1 g_1 + n_2 g_2 + (n_1 - 1)(n_2 - 1).$$

**Proof of Theorem 3.6.** By Theorem 3.2, the curve $X^{\text{pre}} (2, a)$ has genus zero, so it is isomorphic to $\mathbb{P}^1$. The composition

$$\delta_N \circ \cdots \circ \delta_3: X^{\text{pre}} (N, a) \to X^{\text{pre}} (2, a) \cong \mathbb{P}^1$$

has degree $2^{N-2}$, so the gonality of $X^{\text{pre}} (N, a)$ is at most $2^{N-2}$. We prove equality by induction on $N$. Since this is clear for $N = 2$, we may assume that $X^{\text{pre}} (N - 1, a)$ has gonality $2^{N-3}$. Let $\phi: X^{\text{pre}} (N, a) \to \mathbb{P}^1$ be a non-constant
morphism of minimal degree. If \( \phi \) factors through the map \( \delta_N \), then \( \deg \phi \) is twice the gonality of \( X^{\text{pre}} (N-1, a) \), as desired. So assume \( \phi \) does not factor through \( \delta_N \). Since \( \delta_N \) has degree 2, it follows that the map

\[
(\delta_N, \phi) : X^{\text{pre}} (N, a) \to X^{\text{pre}} (N-1, a) \times \mathbb{P}^1
\]

is generically injective, and now Castelnuovo’s inequality implies that

\[
g(N) \leq 2g(N - 1) + (2 - 1)(\deg \phi - 1)
\]

\[
(N - 3)2^{N-2} + 1 \leq 2 \left( (N-4)2^{N-3} + 1 \right) + \deg \phi - 1
\]

\[
2^{N-2} \leq \deg \phi.
\]

Thus the gonality of \( X^{\text{pre}} (N, a) \) is \( \deg \phi = 2^{N-2} \).

**Corollary 3.8.** Let \( a \in K \), and let \( N \geq 3 \) be an integer for which \( Y^{\text{pre}} (N, a) \) is nonsingular. Then \( 2^{N-3} \) is the minimal degree of any nonconstant morphism from \( X^{\text{pre}} (N, a) \) to a genus one curve.

**Proof.** Since the gonality of \( X^{\text{pre}} (N, a) \) is \( 2^{N-2} \), and any genus one curve admits a degree-2 map to \( \mathbb{P}^1 \), any nonconstant morphism from \( X^{\text{pre}} (N, a) \) to a genus-1 curve has degree at least \( 2^{N-3} \). Conversely, this degree occurs for the map

\[
\delta_N \circ \cdots \circ \delta_4 : X^{\text{pre}} (N, a) \to X^{\text{pre}} (3, a).
\]

4. ARITHMETIC OF PRE-IMAGES

Let \( K \) be a number field. For \( a, c \in K \), we are interested in the size of

\[
\{ x_0 \in K : f_c^N(x_0) = a \text{ for some } N \geq 1 \},
\]

the set of pre-images of \( a \) under iterates of \( f_c \). These sets can be arbitrarily large if we allow \( a \) to vary (even if \( c \) is fixed). Indeed, if we choose \( b \in K \) to be a non-preperiodic point for \( f_c \), and put \( a = f_c^N(b) \), then the above set contains (at least) the \( N \) elements \( b, f_c(b), \ldots, f_c^{N-1}(b) \). In this section we show that the situation is different if we fix \( a \) and allow \( c \) to vary.

In particular, we prove Theorem 1.2. To illustrate the method, we begin by proving the following special case (in which no values \( a \) need to be excluded):

**Theorem 4.1.** Let \( K \) be a number field, and pick \( a \in K \). There is an integer \( \nu(K, a) \) such that any \( c \in K \) satisfies

\[
\# \{ x_0 \in K : f_c^N(x_0) = a \text{ for some } N \geq 1 \} \leq \nu(K, a).
\]

**Proof.** Suppose \( M > 0 \) is chosen so that \( Y^{\text{pre}} (M, a) (K) \) is finite. For each \( c \in K \), we must bound the union of the following two sets:

\[
U_c := \{ x_0 \in K : f_c^N(x_0) = a \text{ for some } N < M \}
\]

\[
V_c := \{ x_0 \in K : f_c^N(x_0) = a \text{ for some } N \geq M \}.
\]

For fixed \( c \) and \( N \), the polynomial \( f_c^N(z) \) has degree \( 2^N \), so \( \# U_c \leq \sum_{N=1}^{M-1} 2^N = 2^M - 2 \). If \( V_c \) is nonempty, so \( f_c^N(x_0) = a \) for some \( N \geq M \) and \( x_0 \in K \), then \((f_c^{N-M}(x_0), c) \in Y^{\text{pre}} (M, a)(K) \). Hence there are only finitely many \( c \in K \) for which \( \# V_c > 0 \), and for each such \( c \) the following lemma shows that \( V_c \) is finite. Letting \( S \) be the maximum value of \( \# V_c \), it follows that \( \# (U_c \cup V_c) \leq 2^M - 2 + S \).

It remains to prove that \( Y^{\text{pre}} (M, a) (K) \) is finite for some \( M \). If \( Y^{\text{pre}} (4, a) \) is nonsingular, then \( X^{\text{pre}} (4, a) \) has genus 5 by Theorem 3.2. We apply the Mordell
conjecture (Faltings’ theorem) to conclude that $X_{\text{pre}}(4, a)(K)$ is finite. This implies that $Y_{\text{pre}}(4, a)(K)$ is finite, so we may take $M = 4$. If $Y_{\text{pre}}(4, a)$ is singular and $a \neq -1/4$, then (as noted in Table 3.4) $Y_{\text{pre}}(4, a)$ is geometrically irreducible of genus more than 1, so again Faltings’ theorem implies $Y_{\text{pre}}(4, a)(K)$ is finite. Finally, if $a = -1/4$ then (again from Table 3.4) the set $Y_{\text{pre}}(5, a)$ has two geometrically irreducible components, both of genus 5, so again Faltings’ theorem implies $Y_{\text{pre}}(5, a)(K)$ is finite. Thus, for each $a \in K$, we have exhibited an integer $M$ for which $Y_{\text{pre}}(M, a)(K)$ is finite, and the proof is complete. □

Lemma 4.2. Let $a, c$ be elements of a number field $K$. For any integer $B$, the set

$$\{ x_0 \in \mathbb{Q} : [K(x_0) : K] \leq B \text{ and } f_c^N(x_0) = a \text{ for some } N \geq 1 \}$$

is finite.

Proof. We use standard properties of canonical heights of morphisms, which can be found for instance in [10, §3.4]. The canonical height function $\hat{h}$ associated to $f_c$ satisfies the properties

$$\begin{align*}
\hat{h}(z) &\geq 0 \\
\hat{h}(f_c(z)) &\geq 2\hat{h}(z) \\
\hat{h}(z) &= \hat{h}(z) + O(1)
\end{align*}$$

for all $z \in \mathbb{Q}$, where $h$ is the absolute logarithmic Weil height and the implied constant depends only on $c$.

If $f_c^N(x_0) = a$ for some $N \geq 1$, then

$$h(x_0) = \hat{h}(x_0) + O(1) = 2^{-N}\hat{h}(a) + O(1) \leq \hat{h}(a) + O(1) = h(a) + O(1).$$

In particular, the set described in the lemma is a collection of algebraic numbers of bounded height and degree, and so is finite (for instance by [10, Thm. 3.7]). □

The proof of Theorem 1.2 follows the same strategy as that of Theorem 4.1, but instead of Faltings’ theorem we use a consequence of a more powerful theorem due to Vojta. We need some notation to state this consequence.

If $\phi : C \to C'$ is a non-constant morphism of smooth projective curves with ramification divisor $R_\phi$, define

$$\rho(\phi) = \frac{\deg R_\phi}{2\deg \phi}.$$ 

Theorem 4.3 (Song–Tucker–Vojta). If $\phi : C \to C'$ is a non-constant morphism of smooth projective curves defined over a number field $K$, then the set

$$\Gamma(C, \phi) = \{ P \in C(\overline{\mathbb{Q}}) : [K(P) : K] < \rho(\phi) \text{ and } K(\phi(P)) = K(P) \}$$

is finite.

Vojta proved this result in case $C' = \mathbb{P}^1$ (see [14, Cor. 0.3] and [13, Thm. A]), as a consequence of a deep inequality on arithmetic discriminants. Song and Tucker [11, Prop. 2.3] generalized Vojta’s proof to deduce Theorem 4.3 for arbitrary $C'$. Note that Theorem 4.3 implies the Mordell conjecture: if $C$ has genus at least 2, then any non-constant morphism $\phi : C \to \mathbb{P}^1$ satisfies $\rho(\phi) > 1$, so the finite set $\Gamma(C, \phi)$ includes $C(K)$. 

Remark 4.4. We advise the reader of some typographical errors in [11]. Specifically, the inequality \( \geq \) in [11, Cor. 2.1] should be a strict inequality \( > \), the displayed equality in [11, Rem. 2.4] should say \( \deg R_f = (2g - 2) - (2g' - 2) \deg f \), and the inequality \( > \) in the next line should be \( < \).

We will apply Theorem 4.3 to composite maps of the form \( \delta_M \circ \delta_{M+1} \circ \cdots \circ \delta_{M+J} \).

First we give a consequence of Theorem 4.3 for arbitrary composite maps.

**Lemma 4.5.** Let
\[ X_N \xrightarrow{\phi_N} X_{N-1} \xrightarrow{\phi_{N-1}} \cdots \xrightarrow{\phi_2} X_2 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_0} X_0 \]
be a sequence of smooth projective curves defined over a number field \( K \), equipped with non-constant \( K \)-morphisms \( \phi_M : X_M \to X_{M-1} \) for each \( 1 \leq M \leq N \), and put
\[ B_N := \min_{1 \leq M \leq N} 2^{N-M} \rho(\phi_M) \]
\[ b_N := \min_{1 \leq M \leq N} \rho(\phi_M). \]

Then the set
\[ \{ P \in X_N(\overline{K}) : [K(P) : K] < B_N \text{ and } [K(\phi_1 \circ \cdots \circ \phi_N(P)) : K] \geq b_N \} \]
is finite.

**Proof.** By Theorem 4.3, for each \( M \) with \( 1 \leq M \leq N \) the set
\[ \Gamma(M) := \{ P \in X_M(\overline{K}) : [K(P) : K] < \rho(\phi_M) \text{ and } K(P) = K(\phi_M(P)) \} \]
is finite. For \( 1 \leq M \leq N \), define \( \psi_M : X_N \to X_{N-M} \) by
\[ \psi_M := \phi_{N-M+1} \circ \phi_{N-M+2} \circ \cdots \circ \phi_N, \]
and let \( \psi_0 \) be the identity on \( X_N \). Since \( \psi_M \) is a finite morphism,
\[ \Gamma := \bigcup_{M=0}^N \{ P \in X_N(\overline{K}) : \psi_M(P) \in \Gamma(N-M) \} \]
is a finite union of finite sets, and so is finite. We will show that if \( P \in X_N(\overline{K}) \setminus \Gamma \) satisfies \( [K(\psi_N(P)) : K] \geq b_N \) then \( [K(P) : K] \geq B_N \), which proves that the set defined in (8) is contained in the finite set \( \Gamma \).

Suppose \( P \in X_N(\overline{K}) \setminus \Gamma \) satisfies \( [K(\psi_N(P)) : K] \geq b_N \). Then
\[ K(\psi_N(P)) \subset K(\psi_{N-1}(P)) \subset \cdots \subset K(\psi_0(P)) = K(P). \]
If we choose \( j \) with \( 0 \leq j \leq N - 1 \) and \( \rho(\phi_{N-j}) = b_N \), then
\[ [K(\psi_j(P)) : K] \geq [K(\psi_N(P)) : K] \geq b_N = \rho(\phi_{N-j}). \]
Let \( 0 \leq J \leq N - 1 \) be the least integer such that
\[ [K(\psi_J(P)) : K] \geq \rho(\phi_{N-J}). \]

We may assume \( J \geq 1 \), since otherwise we obtain the desired conclusion
\[ [K(P) : K] = [K(\psi_0(P)) : K] \geq \rho(\phi_N) \geq B_N. \]

By minimality, for \( 0 \leq j < J \) we have
\[ [K(\psi_j(P)) : K] < \rho(\phi_{N-j}); \]
but \( P \notin \Gamma \) implies \( \psi_j(P) \notin \Gamma(N-j) \), so
\[ K(\psi_j(P)) \neq K(\psi_{j+1}(P)). \]
and thus $[K(ψ_2(P)) : K(ψ_{j+1}(P))] \geq 2$. It follows that

$$[K(P) : K] = \left( \prod_{j=0}^{J-1} [K(ψ_2(P)) : K(ψ_{j+1}(P))] \right) [K(ψ_2(P)) : K]$$

$$\geq 2^j \rho(φ_{N-j}) \geq B_N.$$

This completes the proof that the finite set $Γ$ contains the set defined in (8). □

We now prove Theorem 1.3.

Proof of Theorem 1.3. Since the algebraic set $Y^{pre}(3, a)$ has a geometrically irreducible component of genus 0 or 1, there is a finite extension $L$ of $K$ for which $Y^{pre}(3, a)(L)$ is infinite. Since the composite map $ψ := δ_1 δ_2 \cdots δ_N$ defines an endomorphism of $K^2$ of degree $2^{N-3}$, if $ψ(P) ∈ Y^{pre}(3, a)(L)$ then $[L(P) : L] \leq 2^{N-3}$. But $ψ(P) ∈ Y^{pre}(3, a)(\overline{Q})$ if and only if $P ∈ Y^{pre}(N, a)(\overline{Q})$. This proves the first part of Theorem 1.3.

Now suppose $a$ is not a critical value of $f_2^j(0)$ for any $2 ≤ j ≤ N$, so $Y^{pre}(M, a)$ is nonsingular for $M ≤ N$, whence $X^{pre}(M, a)$ is defined. Consider the tower of smooth projective curves

$$X^{pre}(N, a) \overset{δ_N}{\longrightarrow} X^{pre}(N-1, a) \overset{δ_{N-1}}{\longrightarrow} \cdots \overset{δ_2}{\longrightarrow} X^{pre}(1, a),$$

where $δ_M : X^{pre}(M, a) → X^{pre}(M-1, a)$ is the usual map. By Lemma 3.1, the degree of the ramification divisor of $δ_M$ is $2^{M-1}$, so $ρ(δ_M) = 2^{M-3}$. If we apply Lemma 4.5 to this tower of curves, we have (in the notation of that lemma) $B_N = 2^{N-3}$ and $b_N = 1/2$. Theorem 1.3 follows. □

Remark 4.6. By Remark 3.5, the set $Y^{pre}(4, -1/4)(\overline{Q})$ is infinite, so the above proof implies that $Y^{pre}(N, -1/4)(\overline{Q})$ contains infinitely many points of degree at most $2^{N-4}$. Thus, the critical value hypothesis in Theorem 1.3 cannot be removed.

The following refinement of Theorem 1.2 is our main result:

Theorem 4.7 (Uniform Boundedness for Pre-Images). Fix a positive integer $B$, and put $N = [4 + \log_2(B)]$. For any $a ∈ \overline{Q}$ such that $Y^{pre}(N, a)$ is nonsingular, there is an integer $κ(B, a)$ with the following property: for any $c ∈ \overline{Q}$, we have

$$\# \{ x_0 ∈ \overline{Q} : |Q(a, c, x_0) : Q(a)| ≤ B \text{ and } f^M_c(x_0) = a \text{ for some } M ≥ 1 \} ≤ κ(B, a).$$

Moreover, $Y^{pre}(N, a)$ is singular for fewer than $16B$ values $a ∈ \overline{Q}$.

Proof. By Remark 2.3, there are at most $2^N - N - 1$ values $a ∈ \overline{Q}$ for which $Y^{pre}(N, a)$ is singular, which implies the final statement.

Choose $a ∈ \overline{Q}$ such that $Y^{pre}(N, a)$ is nonsingular. For any $c ∈ \overline{Q}$, the set described in the theorem is contained in $U_c \cup V_c$, where

$$U_c := \{ x_0 ∈ \overline{Q} : f^M_c(x_0) = a \text{ for some } M < N \},$$

$$V_c := \{ x_0 ∈ \overline{Q} : |Q(a, c, x_0) : Q(a)| < 2^{N-3} \text{ and } f^M_c(x_0) = a \text{ for some } M ≥ N \}.$$

By Theorem 1.3, there are only finitely many points $(y_0, c_0) ∈ Y^{pre}(N, a)(\overline{Q})$ for which $|Q(a, y_0, c_0) : Q(a)| < 2^{N-3}$. For each such $c_0$, Lemma 4.2 implies $V_{c_0}$ is finite; for any other $c$ we have $V_c = 0$. Letting $S$ be the maximum of $V_c$ over all $c ∈ \overline{Q}$, it follows that $S$ is an integer depending only on $N$ and $a$. Since $f^M_c(z)$ has degree $2^M$, we have $\#U_c < 2^N$, so $\#(U_c \cup V_c) < S + 2^N$. □
Theorem 4.7, as well as several other results in this paper, applies to values $a$ for which a particular $Y^\text{pre}(N,a)$ is nonsingular. We now describe a large class of such values $a$.

**Proposition 4.8.** Let $\mathcal{O}_K$ be the ring of integers in a number field $K$, and let $a \in K$. Suppose $a$ is integral with respect to some prime ideal of $\mathcal{O}_K$ lying over 2; in other words, $a = a_1/a_2$ with $a_1, a_2 \in \mathcal{O}_K$ and $a_2 \notin p$ for some $p \mid 2$. Then $Y^\text{pre}(N,a)$ is nonsingular for every $N \geq 1$.

**Proof.** By Proposition 2.1, it suffices to show there do not exist an integer $2 \leq j \leq N$ and an element $c_0 \in \mathbb{Q}$ for which

$$f_j^{c_0}(0) = a \quad \text{and} \quad \frac{\partial f_j^{c_0}(0)}{\partial c} \bigg|_{c=c_0} = 0.$$ 

Suppose $j$ and $c_0$ satisfy these conditions, and write $P(c) = f^c_j(0) - a \in K[c]$. Letting $R$ be the localization of $\mathcal{O}_K$ at the prime ideal $p$, our hypothesis on $a$ shows that $P$ is a monic polynomial over $R$. Since $P(c_0) = 0$, the ring $R[c_0]$ is integral over $R$, and so contains a prime ideal $q$ lying above $p$.

Writing $P(c) = Q(c)^2 + c - a$ with $Q = f_j^{c_0}(0) \in \mathbb{Z}[c]$, we have $P'(c) = 2Q(c)Q'(c) + 1$. By assumption, $c_0$ is a double root of $P(c)$, and so

$$0 = P'(c_0) = 2Q(c_0)Q'(c_0) + 1.$$ 

Since $Q(c_0)Q'(c_0) \in R[c_0]$, we may reduce this equation modulo $q$ to obtain the contradiction

$$0 \equiv 1 \pmod{q}.$$

Thus $Y^\text{pre}(N,a)$ is nonsingular. $\square$

In particular, this result applies to any algebraic integer $a$, or more generally to any ratio $a = \alpha/m$ with $\alpha$ an algebraic integer and $m$ an odd integer. For such values $a$, we know the genus and gonality of $X^\text{pre}(N,a)$, and moreover we have uniform bounds on the pre-images of $a$ under the various maps $f_c$.

**Remark 4.9.** Our results are related to the study of uniform lower bounds on the canonical height $\hat{h}$ associated to $f_c$, as $c$ varies. A special case of a conjecture of Silverman [10, Conj. 4.98] asserts that, for every number field $K$, there exists a constant $\epsilon = \epsilon(K) > 0$ such that either $\hat{h}(\alpha) = 0$ or $\hat{h}(\alpha) \geq \epsilon \max(1, h(c))$ for each $\alpha, c \in K$. (This is a dynamical analogue of a conjecture of Lang’s on heights of non-torsion rational points on elliptic curves.) If this conjecture were true, we could prove Theorem 4.1 without using Faltings’ theorem, so long as we assume that $a$ is not preperiodic for $f_c$. For such $a$ and $c$, if $f_N^c(x_0) = a$ then $x_0$ is not preperiodic for $f_c$, so $\hat{h}(x_0) \neq 0$ and thus

$$2^N \epsilon \max(1, h(c)) \leq 2^N \hat{h}(x_0) = \hat{h}(a) \leq h(a) + h(c) + \log 2,$$

where the last inequality follows from decomposing the heights into sums of local heights. This bounds $N$ in terms of $K$, $h(a)$, and $\epsilon$; the rest of the proof is as before. Partial results in the direction of Silverman’s conjecture (see [5]) imply an effective version of Theorem 1.2 if the bound $\kappa$ is allowed to depend on the number of primes of $K$ at which $c$ is not integral (in addition to $BD$ and $a$). Of course, this is much weaker than Theorem 1.2, in which $\kappa$ does not depend on $c$. 

In the other direction, since $X^{\operatorname{pre}}(3,0)$ is a rank-one elliptic curve over $\mathbb{Q}$, with unbounded real locus, there are infinitely many $(x_0,c) \in Y^{\operatorname{pre}}(3,0)(\mathbb{Q})$ with $|c| > 4$. For such $(x_0,c)$ we have $f_4^3(x_0) = f_c(0) = c$, so [5, Lemmas 3 and 6] imply
\[ \hat{h}(x_0) = 2^{-4}h(c) \leq \frac{1}{16} h(c) + \frac{\log(5) - 2\log(2)}{16}. \]
Thus, if $\epsilon(\mathbb{Q})$ exists then it is at most $1/16$. A similar construction was given in [5, §5], using the points $(k,-k^2-k+1)$ on $Y^{\operatorname{pre}}(2,-3k^2+2)$ to deduce an upper bound of $1/8$; note that that construction exhibits an infinite family of integral points, whereas each curve $X^{\operatorname{pre}}(2,a)$ has only finitely many such points (since it is a genus zero curve with two rational points at infinity).

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