Chapter 7

Definite Integral

7.4 Calculation of definite integrals (I)

7.4.1 Fundamental formula for calculation of definite integrals

In the previous sections, we evaluated some simple definite integrals using the definition of definite integrals. The calculation needs some special technique even for simple integration. In fact, it does not work well to evaluate a definite integral by taking limit of Riemann sum since it is difficult to give a simple expression to a Riemann sum. In this section, the general methods in calculating definite integrals will be derived by others ways.

According to Property 9 in the last section, $F(x) = \int_a^x f(t) \, dt$ must be continuous over $[a, b]$ if $f(x)$ is an integrable function over $[a, b]$. Now $F(x)$ has a stronger property if $f(x)$ is imposed on a stronger condition than integrable.

Theorem 1 If $f(x)$ is continuous over $[a, b]$, then $G(x) = \int_a^x f(t) \, dt$ is differentiable over $[a, b]$ and

$$G'(x) = f(x)$$

Proof:

$$G(x) = \int_a^x f(t) \, dt, \quad G(x + \Delta x) = \int_a^{x+\Delta x} f(t) \, dt$$
\[ G(x + \Delta x) - G(x) \]
\[ \frac{\Delta x}{\Delta x} \]
\[ = \frac{1}{\Delta x} \int_{x}^{x+\Delta x} f(t) \, dt \]
\[ = \frac{1}{\Delta x} \cdot f(\xi) \Delta x \quad (\exists \xi \in [x, x + \Delta x] \text{ by the intermediate value theorem for the integral}) \]
\[ = f(\xi) \]

On the other hand,

\[ f(x) \text{ is continuous over } [a, b] \]

\[ \Rightarrow \]

\[ \lim_{\Delta x \to 0} f(\xi) = f(x) \quad (\because \xi \in [x, x + \Delta x]) \]

from (7.1)

\[ \lim_{\Delta x \to 0} \frac{G(x + \Delta x) - G(x)}{\Delta x} = \lim_{\Delta x \to 0} f(\xi) = f(x) \]

\[ \Rightarrow \]

\[ G(x) = \int_{a}^{b} f(t) \, dt \text{ is differentiable over } [a, b] \text{ and } G'(x) = f(x) \]

Q.E.D.

Theorem 1 shows that \( \int_{a}^{b} f(t) \, dt \) is an antiderivative of \( f(x) \) if \( f(x) \) is a continuous function over \([a, b]\). In other words, there exists an antiderivative for any continuous function. We have the following important formula:

**Fundamental Theorem:** Let \( f(x) \) be a continuous function over \([a, b]\),
\( F(x) \) an antiderivative of \( f(x) \), i.e. \( F'(x) = f(x) \). Then

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]

**Proof:** According to Theorem 1,

\[ G(x) = \int_{a}^{b} f(t) \, dt \text{ is an antiderivative of } f(x) \]
\[ \int_a^b f(x) \, dx = G(b) - G(a) \quad (\therefore G(a) = \int_a^a f(x) \, dx = 0) \]
\[ = [F(b) - C] - [F(a) - C] \]
\[ = F(b) - F(a) \]

Q.E.D.

\[ F(x) \bigg|_a^b \triangleq F(b) - F(a) \]

The fundamental formula can be rewritten into the following formula, so called as Newton-Leibniz formula.

Newton-Leibniz formula:

\[ \int_a^b f(x) \, dx = F(x) \bigg|_a^b \]

**Example 1**: Finding \( \int_0^{1/2} e^{2x} \, dx \).

**Solution**: Since \( \int e^{2x} \, dx = \frac{1}{2} e^{2x} + C \), then \( F(x) = \frac{1}{2} e^{2x} \) is an antiderivative of \( f(x) = e^{2x} \). From Newton-Leibniz formula,

\[ \int_0^{1/2} e^{2x} \, dx = \frac{1}{2} e^{2 \cdot 1/2} \bigg|_0^{1/2} = \frac{1}{2} \left[ e^{(1/2)} - e^{2(0)} \right] = \frac{1}{2} (e - 1) \]

**Example 2**: Finding \( \int_0^2 \frac{x}{\sqrt{1+x^2}} \, dx \).

**Solution**: Since \( \int \frac{x}{\sqrt{1+x^2}} \, dx = \sqrt{1+x^2} + C \), then \( F(x) = \sqrt{1+x^2} \) is an antiderivative of \( f(x) = \frac{x}{\sqrt{1+x^2}} \). From Newton-Leibniz formula,

\[ \int_0^2 \frac{x}{\sqrt{1+x^2}} \, dx = \sqrt{1+2^2} \bigg|_0^2 = \sqrt{1+4} - \sqrt{1+0^2} = \sqrt{5} - 1 \]
Example 3: A car runs at the speed of 32 kilometer per hour. When arriving at some place, the car needs to slow down to a halt. Assume the car decelerates at the decelerator $a = 1.8 \text{m/s}^2$. What is the distance between the place where the car begins breaking and the place where the car stops?

Solution: Let $v(t)$ be the speed of the car at the time $t$. Suppose $t = 0$ when the car begins breaking. Then

$$v(0) = \frac{32 \text{km/hr}}{9} = \frac{80}{9} \text{m/s}$$

and $v(t) = v(0) - at = \frac{80}{9} - 1.8t$

Let $T$ be the moment when $v(T) = 0$, then

$$\frac{80}{9} - 1.8T = 0 \implies T = \frac{400}{81} \text{ (s)}$$

So the distance between the place where the car begins breaking and the place where the car stops is:

$$S = \int_0^T v(t) \, dt = \int_0^{400/81} \left( \frac{80}{9} - 1.8t \right) \, dt = \left( \frac{80}{9}t - 0.9t^2 \right) \bigg|_0^{400/81} = 21.9 \text{ m}$$

7.4.2 Change of variable formula for definite integrals

Theorem 2: Let $f(x)$ be continuous over $[a, b]$. Let $x = \varphi(t)$, where $\varphi(t)$ has a continuous derivative $\varphi'(t)$ over $[\alpha, \beta]$ and $\alpha \leq \varphi(t) \leq \beta$ for $\alpha \leq t \leq \beta$, $\varphi(\alpha) = a$, $\varphi(\beta) = b$. Then

$$\int_a^b f(x) \, dx = \int_\alpha^\beta f(\varphi(t)) \varphi'(t) \, dt \quad (7.2)$$

Proof: $f(x)$ is continuous over $[a, b]$, $\varphi(t)$ and $\varphi'(t)$ are continuous over $[\alpha, \beta]$, so $f(x)$ and $f(\varphi(t)) \varphi'(t)$ are integrable over $[a, b]$ and $[\alpha, \beta]$, respectively.

Let $G(x)$ be an antiderivative of $f(x)$, then $G[\varphi(t)]$ is an antiderivative of $f(\varphi(t)) \varphi'(t)$. From Newton-Leibniz formula,

$$\int_a^b f(x) \, dx = G(b) - G(a)$$

$$\int_\alpha^\beta f(\varphi(t)) \varphi'(t) \, dt = G[\varphi(\beta)] - G[\varphi(\alpha)]$$

$$= G(b) - G(a)$$
\[
\int_a^b f(x) \, dx = \int_0^\beta f(\varphi(t)) \varphi'(t) \, dt
\]
Q.E.D.

**Example 4:** Finding \( \int_0^a \sqrt{a^2 - x^2} \, dx \)

**Solution:** Let \( x = a \cdot \sin t \), i.e. \( \varphi(t) = a \cdot \sin t \). Since
\[
\varphi(0) = 0, \quad \varphi\left(\frac{\pi}{2}\right) = a \quad \text{and} \quad \varphi'(t) = a \cdot \cos t
\]

\[
\int_0^a \sqrt{a^2 - x^2} \, dx = \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2 t} \left| a \cdot \cos t \right| \, dt
\]
\[
= a^2 \int_0^{\pi/2} \cos^2 t \, dt
\]
\[
= \frac{a^2}{2} \left( t + \frac{1}{2} \sin 2t \right) \bigg|_0^{\pi/2}
\]
\[
= \frac{\pi a^2}{4}
\]

**Example 5:** Finding \( \int_a^{2a} \frac{\sqrt{2^2 - a^2}}{x^2} \, dx \)

**Solution:** Let \( x = a \cdot \sec t \), i.e. \( \varphi(t) = a \cdot \sec t \). Since
\[
\varphi(0) = 1, \quad \varphi\left(\frac{\pi}{3}\right) = 2a \quad \text{and} \quad \varphi'(t) = a \cdot \frac{\sin t}{\cos^2 t}
\]

\[
\int_a^{2a} \frac{\sqrt{x^2 - a^2}}{x^2} \, dx = \int_0^{\pi/3} \frac{\sqrt{a^2 \sec^2 t - a^2}}{a^4 \sec^4 t} \left( a \cdot \frac{\sin t}{\cos^2 t} \right) \, dt
\]
\[
= \frac{1}{a^2} \int_0^{\pi/3} \sin^2 t \cdot \cos t \, dt
\]
\[
= \frac{1}{a^2} \left[ \frac{\sin^3 t}{3} \right]_0^{\pi/3}
\]
\[
= \frac{\sqrt{3}}{8a^2}
\]
Example 6 Finding $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} \, dx$

Solution: $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} \, dx = \int_{\pi/2}^\pi \frac{x \sin x}{1 + \cos^2 x} \, dx + \int_0^{\pi/2} \frac{x \sin x}{1 + \cos^2 x} \, dx$. For the second integration, let $x = \pi - t$, i.e. $\varphi(t) = \pi - t$, then

$$\int_{\pi/2}^\pi \frac{x \sin x}{1 + \cos^2 x} \, dx = -\int_{\pi/2}^\pi \frac{(\pi - t) \sin t}{1 + \cos^2 t} \, dt$$

$$= \int_0^{\pi/2} \frac{\pi \sin t}{1 + \cos^2 t} \, dt - \int_0^{\pi/2} \frac{t \sin t}{1 + \cos^2 t} \, dt$$

$$= \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} \, dx - \int_0^{\pi/2} \frac{x \sin x}{1 + \cos^2 x} \, dx$$

$$\Rightarrow$$

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} \, dx = \int_0^{\pi/2} \frac{x \sin x}{1 + \cos^2 x} \, dx + \left[ \int_0^{\pi/2} \frac{\pi \sin x}{1 + \cos^2 x} \, dx - \int_0^{\pi/2} \frac{x \sin x}{1 + \cos^2 x} \, dx \right]$$

$$= \int_0^{\pi/2} \frac{x \sin x}{1 + \cos^2 x} \, dx$$

$$= -\pi \arctg (\cos t) \bigg|_0^{\pi/2}$$

$$= \frac{\pi^2}{4}$$

7.4.3 Formula for integration by parts for the definite integral

$$(uv)' = u'v + uv' \Rightarrow$$

$uv$ is an antiderivative of $u'v + uv'$. So

$$\int_a^b (u'v + uv') \, dx = uv \bigg|_a^b \Rightarrow$$

$$\int_a^b u' \, dx = uv \bigg|_a^b - \int_a^b u' \, dx$$

Homework: Page 302-303, 1(4), 1(9), 1(11), 1(15); 2(3), 2(4), 2(5), 4