Chapter 7

Definite Integral

7.4 Calculation of definite integrals (III)

7.4.5 Some other properties and extra examples of definite integrals

Property 1 (Integration of periodic functions): Let \( f(x) \) be a periodic function with period \( T \), then

\[
\int_{a}^{a+nT} f(x) \, dx = n \int_{0}^{T} f(x) \, dx
\]

where \( n \) is any natural number. (P. 303, Ex. 3)

Proof:

\( T \) is a period of \( f(x) \)

\[ \implies \]

\( kT \) is also a period of \( f(x) \), \( k = 1, 2, 3, \ldots, n \)

and \( \int_{kT}^{(k+1)T} f(x) \, dx \) Let \( x = t + kT \)

\[
\int_{0}^{T} f(t + kT) \, dt = \int_{0}^{T} f(t) \, dt
\]

Example 10

\[
\int_{a}^{a+nT} f(x) \, dx = n \int_{0}^{T} f(x) \, dx
\]
\[
\begin{align*}
= & \int_0^T f(x) \, dx + \int_T^{2T} f(x) \, dx + \ldots + \int_{(n-1)T}^{nT} f(x) \, dx \\
= & \underbrace{\int_0^T f(x) \, dx + \int_0^T f(x) \, dx + \ldots + \int_0^T f(x) \, dx}_{n} \\
= & n \int_0^T f(t) \, dt
\end{align*}
\]

Q.E.D.

**Property 2:** Any antiderivative of an odd function must be an even function; one of the antiderivatives of an even function must be an odd function. (P. 304, Ex. 6)

**Proof:** Without lose of the generality, we assume that the domains of the functions considered in this property are the whole \( x \)-axis. (1) Let \( f(x) \) be an odd function, then

\[ f(-x) = -f(x) \]

and any antiderivative function of \( f(x) \) can be expressed as

\[ F(x) = \int_0^x f(t) \, dt + C, \quad \text{where } C \text{ is any constant.} \]

\[ \Rightarrow \]

\[ F(-x) = \int_0^{-x} f(t) \, dt + C \\
= \int_0^x f(-s) \, (-ds) + C \quad \text{(let } s = -t) \\
= \int_0^x f(s) \, ds + C = F(x) \]

\[ \Rightarrow \]

\( F(x) \) is an even function.

(2) Let \( g(x) \) be an even function, then

\[ g(-x) = g(x) \]

and

\[ G(x) = \int_0^x g(t) \, dt \text{ is an antiderivative function of } g(x) \]
Since
\[ G(-x) = \int_0^{-x} g(t) \, dt \]
\[ = \int_0^{x} g(-s)(-ds) \quad \text{(let } s = -t) \]
\[ = -\int_0^{x} g(s) \, ds = G(x) \]

i.e.,
\[ G(x) \text{ is an odd function.} \]

Q.E.D.

**Property 3:** If the function \( f(x) \) is symmetric about the line \( x = T \), and \( a < T < b \). Then
\[ \int_a^b f(x) \, dx = 2 \int_T^b f(x) \, dx + \int_a^{2T-b} f(x) \, dx \]

(P. 304, Ex. 7)

**Proof:** By assumption \( f(2T - x) = f(x) \). First we have
\[ \int_a^b f(x) \, dx = \int_a^{2T-b} f(x) \, dx + \int_T^b f(x) \, dx, \]

and
\[ \int_T^b f(x) \, dx = \int_T^{2T-b} f(x) \, dx + \int_T^b f(x) \, dx. \]

Furthermore
\[ \int_T^{2T-b} f(x) \, dx = \int_T^{2T-b} f(2T - x) \, dx \]
\[ = -\int_T^b f(x) \, dx \]
\[ = \int_T^b f(x) \, dx. \]

So
\[ \int_T^{2T-b} f(x) \, dx = 2 \int_T^b f(x) \, dx. \]
Then it follows
\[ \int_a^b f(x) \, dx = 2 \int_a^b f(x) \, dx + \int_a^{2T-b} f(x) \, dx \]

**Remark 1** Geometric explanation:

(i) If $2T-b \geq a$ then $\int_a^b f(x) \, dx$ is the sum of $\int_a^{2T-b} f(x) \, dx$ and $2 \int_T^b f(x) \, dx$ because of the symmetry of $f$ about the line $x = T$.

(ii) If $2T-b < a$ then $\int_a^b f(x) \, dx$ is $2 \int_T^b f(x) \, dx$ minus $\int_{2T-b}^a f(x) \, dx$, i.e. $\int_a^b f(x) \, dx = 2 \int_T^b f(x) \, dx + \int_a^{2T-b} f(x) \, dx$.

Q.E.D.

**Property 4 (Cauchy-Schwartz inequality):** Suppose that the functions $f(x)$ and $g(x)$ are integrable, then
\[ \left( \int_a^b f(x) g(x) \, dx \right)^2 \leq \int_a^b f^2(x) \, dx \cdot \int_a^b g^2(x) \, dx \]

In addition, suppose that $f(x)$ and $g(x)$ are continuous, then the equality holds if and only if there exists a constant $k$ such that $f(x) = k \cdot g(x)$.

**Proof:** Let $\lambda$ be any constant, then $|f(x) - \lambda g(x)|^2 \geq 0$. So
\[ 0 \leq \int_a^b [f(x) - \lambda g(x)]^2 \, dx \tag{7.1} \]
\[ = \int_a^b f^2(x) \, dx - 2\lambda \int_a^b f(x) g(x) \, dx + \lambda^2 \int_a^b g^2(x) \, dx \]

Since (7.1) holds for any real $\lambda$, the discriminant $\Delta$ for $\lambda^2 \int_a^b g^2(x) \, dx$ must be non-positive, i.e. $\Delta = 4 \left( \int_a^b f(x) g(x) \, dx \right)^2 - \left( \int_a^b g^2(x) \, dx \right) \left( \int_a^b f^2(x) \, dx \right)$ must be non-positive. This is because that
\[ \int_a^b f^2(x) \, dx - 2\lambda \int_a^b f(x) g(x) \, dx + \lambda^2 \int_a^b g^2(x) \, dx \]
\[ = \left( \int_a^b g^2(x) \, dx \right) (\lambda - \frac{\int_a^b f(x) g(x) \, dx}{\int_a^b g^2(x) \, dx})^2 - \frac{\Delta}{4 \int_a^b g^2(x) \, dx} \geq 0 \text{ for any } \lambda \]
(especially for $\lambda = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g^2(x)dx}$ and $\int_a^b g^2(x)dx \geq 0$. Then $\Delta \leq 0$ implies

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leq \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx$$

Discussion about the equality:
If $f(x) = \lambda g(x)$, obviously the equality holds. But the equality might still hold even if $f(x) \neq \lambda g(x)$ when we drop the condition of the continuity of $f$ and $g$. For example, for

$$f(x) = \begin{cases} g(x) & x \neq x_0 \\ g(x_0) + 1 & x = x_0 \end{cases}$$

$f(x) \neq g(x)$, but the equality holds.

Now we assume that $f(x)$ and $g(x)$ are continuous and the equality holds. Then we can prove that there exists a constant $\lambda$ such that $f(x) = \lambda g(x)$. In fact, if $f(x) \neq \lambda g(x)$ for any constant $\lambda$, then

$$\int_a^b [f(x) - \lambda g(x)]^2 dx > 0$$

especially, for $\lambda = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g^2(x)dx}$, we have $\int_a^b \left| f(x) - \lambda g(x) \right|^2 dx > 0$ (if $g(x)$ is continuous and $g(x)$ is not identically 0, then $\int_a^b \left| g(x) \right|^2 dx > 0$).

This implies

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leq \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx.$$  

This contradicts the equality $\left[\int_a^b f(x)g(x)dx\right]^2 = \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx$.

Q.E.D.

Example 1: Determining the integrability of the following functions over $[0, 2]$, and then estimating the definite integrals of those integrable functions over $[0, 2]$.

1. $f(x) = \begin{cases} x, & 0 \leq x < 1 \\ x - 2, & 1 \leq x \leq 2 \end{cases}$; (Answer: $\int_0^2 f(x)dx = \int_0^1 xdx + \int_1^2 (x - 2)dx = 0$)
2. \( f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ x - 2, & 1 < x \leq 2 \end{cases} \) (Answer: \( \int_0^2 f(x) \, dx = \int_0^1 x \, dx + \int_1^2 (x - 2) \, dx = 0 \))

3. \( f(x) = x + [x] \); (Answer: \( \int_0^2 f(x) \, dx = \int_0^1 x \, dx + \int_1^2 (x + 1) \, dx = 1/2 + 5/2 = 3 \))

4. \( f(x) = \begin{cases} x + [x], & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases} \)

5. \( f(x) = \begin{cases} 1, & x = a + b\sqrt{2} \text{ where } a \text{ and } b \text{ are rational} \\ 0, & x \neq a + b\sqrt{2} \text{ for any rational number } a \text{ and } b \end{cases} \)

6. \( f(x) = \begin{cases} \frac{1}{[x]}, & 0 < x \leq 1 \\ 0, & x = 0 \text{ or } x > 1 \end{cases} \)

7. the following function:

![Graph](attachment:image.png)

**Solution:** (4) We just consider the integrality of \( f(x) \) over \([0, 1]\), so

\[
 f(x) = \begin{cases} x, & x \text{ is rational number in } [0, 1) \\ x + 1, & x = 1 \\ 0, & x \text{ is irrational number in } [0, 1] \end{cases}
\]
7.4. CALCULATION OF DEFINITE INTEGRALS (III)

Give a portion $P$ of $[0, 1]$: $P_n = \{\frac{a}{n}, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$, the Darboux upper sum and lower sum are, respectively,

$$\bar{\sigma}(f, P_n) = \sum_{k=1}^{n-1} \frac{k}{n} \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} = \frac{1}{2} - \frac{1}{2n} + \frac{2}{n} = \frac{n+3}{2n}$$

$$\underline{\sigma}(f, P_n) = \sum_{k=1}^{n} 0 \cdot \frac{1}{n} = 0$$

$$\Rightarrow \lim_{n \to \infty} \bar{\sigma}(f, P_n) = \frac{1}{2} \neq 0 = \lim_{n \to \infty} \underline{\sigma}(f, P_n)$$

$$\Rightarrow f(x) \text{ is not integrable over } [0, 1], \text{ then not integrable over } [0, 2]$$

(5) Since rational numbers are dense in $[0, 2]$, so for any $x \in [0, 2]$ and $\varepsilon > 0$, there exist rational numbers $a$ and $b$ such that

$$|a - x| < \frac{\varepsilon}{2} \text{ and } |b| < \frac{\varepsilon}{2\sqrt{2}}$$

$$\Rightarrow |(a + b\sqrt{2}) - x| \leq |a - x| + |b\sqrt{2}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow$$

The set $A = \{a + b\sqrt{2} : a \text{ and } b \text{ are rational,} \}$ is dense in $[0, 2]$.

On the other hand, we will prove that $B = [0, 2] \setminus A$ is also dense in $[0, 2]$. In fact, the set $C = \{a\sqrt{3} : a \text{ is rational}\} \setminus \{0\}$ is dense in $[0, 2]$ and $C \cap A = \emptyset$. So $C \subset B$ and then

for any portion $P$ to $[0, 2]$, \[ \begin{cases} \text{Darboux upper sum } \bar{\sigma}(f, P) = 1 \\ \text{Darboux lower sum } \underline{\sigma}(f, P) = 0 \end{cases} \]

$$\Rightarrow f(x) \text{ is not integrable over } [0, 2]$$

(6) $n \leq \frac{1}{a} < n + 1 \iff \frac{1}{n} \geq a > \frac{1}{n+1} \Rightarrow \frac{1}{a} = \frac{1}{n}$

So

$$f(x) = \begin{cases} \frac{1}{n} & a \geq x > \frac{1}{n+1} \\ 0 & x = 0 \text{ or } x > 1 \end{cases}$$
\[
\int_0^2 f(x) \, dx = 1 \cdot (1 - \frac{1}{2}) + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} + \cdots + \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n+1} + \cdots \\
= \sum_{n=1}^{\infty} \frac{1}{n^2} - (1 - \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} + \cdots + \frac{1}{n} \cdot \frac{1}{n+1} + \cdots) \\
= \sum_{n=1}^{\infty} \frac{1}{n^2} - (1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} + \cdots) \\
= \frac{\pi^2}{6} - 1
\]

(7) \( f(x) \) is continuous on (0, 2], so it is integrable over [0, 2].

\[
\int_0^2 f(x) \, dx = \int_1^2 f(x) \, dx + \int_{1/2}^1 f(x) \, dx + \cdots + \int_{1/2^{k-1}}^{1/2^k} f(x) \, dx + \cdots \\
= \frac{1}{2} \cdot (2 - 1) + \frac{1}{2} \cdot \frac{1}{2} \cdot (1 - \frac{1}{2}) + \cdots + \frac{1}{2} \cdot \left( \frac{1}{2^k-1} - \frac{1}{2^k} \right) + \cdots \\
= \frac{1}{2} \lim_{k \to \infty} \left( 2 - \frac{1}{2^k} \right) = 1
\]

Example 2: If \( f(x) \) is integrable over \([a, b]\), then there exists \( x \in [a, b] \) such that \( \int_a^x f(t) \, dt = \int_x^b f(t) \, dt \). In addition, this \( x \) is not always in the open interval \((a, b)\).

Proof: Let \( g(x) = \int_a^x f(t) \, dt - \int_x^b f(t) \, dt \), then

\[
g(x) \text{ is continuous and } g(a) = -\int_a^b f(x) \, dx, g(b) = \int_a^b f(x) \, dx
\]

\[
\Rightarrow
\]

\[
\exists x \in [a, b], \text{ s.t. } g(x) = 0, \text{ i.e. } \int_a^x f(t) \, dt = \int_x^b f(t) \, dt
\]

This \( x \) is not always in the open interval \((a, b)\). In fact, let

\[
f(x) = \begin{cases} 
2x, & 0 \leq x < 1/2 \\
2 - 2x, & 1/2 \leq x < 3/2 \\
2x - 4, & 3/2 \leq x \leq 2
\end{cases}
\]
We can just take $x = a$ or $x = b$ such that $\int_a^x f(t) \, dt = \int_x^b f(t) \, dt$. 