Chapter 8

The application and the approximate computation of definite integral

8.1 The area of planar figure (area between curves)

In the last chapter we know that given a non-negative function \( f(x) \) the area bounded by the graph of \( f; x \)-axis and the verticals lines \( x = a; x = b \) is

\[
\int_{a}^{b} f(x) \, dx.
\] (8.1)

In this chapter we'll use integrals to find areas of regions that lie between the graphs of two functions.

Consider the region \( S \) that lies between two curves \( y = f(x) \) and \( y = g(x) \) and between the verticals lines \( x = a; x = b \) where \( f \) and \( g \) are continuous functions and \( f(x) \geq g(x) \) for all \( x \) in \([a; b]\). Then the area of \( S \) is

\[
\int_{a}^{b} [f(x) - g(x)] \, dx.
\] (8.2)

Notice that in the special case where \( g(x) = 0; S \) is the region under the graph of \( f \) and our general definition of the area (8.2) reduces to (8.1). Actually we have

The area of \( S \) = area under \( y = f(x) \) - area under \( y = g(x) \)
CHAPTER 8. THE APPLICATION AND THE APPROXIMATE COMPUTATION OF \( \int_a^b f(x) \, dx \)

\[
\int_a^b f(x) \, dx = \int_a^b g(x) \, dx = \int_a^b [f(x) - g(x)] \, dx.
\]

Example 1 Find the area of the region bounded below by the parabola \( y^2 = 2px \); and bounded above by the parabola \( x^2 = 2py \):

Solution: For such kind of problems, we need to sketch the region enclosed by the curves and determine which curve is the above one and which is the bottom one and then find the points of intersection of the curves by solving their equations simultaneously. The points of intersection of the parabola \( y = \frac{1}{2}p x \) (\( y = i \); \( 2px \) is discarded because it does not get involved the region of interest,) and the parabola \( y = \frac{x^2}{2p} \); is \((0,0)\) and \((2p, 2p)\). So the area

\[
A = \int_0^{2p} \left( \frac{1}{2}px - \frac{x^2}{2p} \right) \, dx = \frac{4}{3}p^2
\]

8.1.1 Area between parametrized curves

If the curve is given by parametric equations \( x = x(t); y = y(t) \) and is traversed once as \( t \) increases from \( \alpha \) to \( \beta \); then we can adapt \( \int_a^b f(x) \, dx \) by using the Substitution Rule for Definite Integrals as follows:

\[
A = \int_0^{2p} \left( \frac{1}{2}px - \frac{x^2}{2p} \right) \, dx = \frac{4}{3}p^2
\]

Example 2 Find the area \( A \) of the ellipse

\[
\begin{align*}
x &= a \cos t \\
y &= b \sin t
\end{align*}
\]

(a > 0; b > 0)

Solution: Because of the symmetry of the ellipse about the \( x \)-axis and \( y \)-axis we just need find the area of the ellipse in the 1st quadrant, i.e.

\[
\frac{A}{4} = \int_0^a y \, dx
\]
8.1. THE AREA OF PLANAR FIGURE (AREA BETWEEN CURVES)

\[ Z_0^{\frac{\pi}{2}} (b \sin t)(a \sin t) dt \]
\[ = Z_{\frac{\pi}{4}}^{\frac{\pi}{2}} \]
\[ = ab \int_0^{\frac{\pi}{2}} \sin^2 t dt \]
\[ = \frac{1}{4} ab \]

So \( A = \frac{1}{4} ab \)

8.1.2 Area in polar coordinates

Suppose that the curve is given by a polar equation \( r = r(\mu) \)

Now we want to find the area of the region \( R \) bounded by polar curve \( r = r(\mu) \) and two rays \( \mu = a; \mu = b \) where \( r(\mu) \) is continuous and where \( 0 < b - a < 2\pi \). We divide the interval \( [a; b] \) into \( n \) subintervals with endpoints \( \mu_0; \mu_1; \ldots; \mu_n \); the rays \( \mu = \mu_i \) then divide \( R \) into \( n \) smaller regions with central angle \( \theta = \mu_i - \mu_{i-1} \). If we choose \( \mu_i \) in the \( i \)-th subintervals \( [\mu; \mu_i] \); then the area \( A_i \) of the \( i \)-th region is approximated by the area of the sector of a circle with the central angle and radius \( r(\mu_i) \); i.e.

\[ A_i \approx \frac{1}{2} \left[ r(\mu_i) \right]^2 \theta \mu_i \]

and an approximate to the total area \( A \) of \( R \) is

\[ A \approx \frac{1}{2} \sum_{i=1}^{n} \left[ r(\mu_i) \right]^2 \theta \mu_i \]

It can proved that the area of \( R \) is

\[ A = \int_a^b \frac{1}{2} [r(\mu)]^2 d\mu \]

When we use this formula it is helpful to think of the area as being swept out by the rotating rays through \( O \) that start with angle \( a \) and ends with angle \( b \).

If the region is bounded two polar curves

\[ r = r_1(\mu); r = r_1(\mu) \cdot r_2(\mu) \cdot r_3(\mu) \]
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and two rays $\mu = a; \mu = b$ the area of the region is

$$A = \int_{a}^{b} \frac{1}{2} r_1^2(\mu) + \int_{a}^{b} \frac{1}{2} r_2^2(\mu) d\mu$$

Example 3 Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\mu$.

Solution: The region enclosed by the right loop is swept out by a ray that rotates from $\mu = \frac{1}{4}$ to $\mu = \frac{1}{4}$: So

$$A = \int_{\frac{1}{4}}^{\frac{1}{4}} \frac{1}{2} r_1^2 d\mu = \frac{1}{2} \int_{\frac{1}{4}}^{\frac{1}{4}} \cos^2 2\mu d\mu$$

$$= \frac{1}{2} \left[ \frac{1}{2} (1 + \cos 4\mu) \right]_{\frac{1}{4}}^{\frac{1}{4}} = \frac{1}{4}$$

8.3 Arc length

Now suppose that curve $C$ with two endpoints $A$ and $B$ is defined by the parametric equations

$$x = x(t); y = y(t); \quad (8.3)$$

$(x(0); y(0)) = A; (x(\frac{1}{4}); y(\frac{1}{4})) = B$ and $x'(t)$ and $y'(t)$ are continuous. We obtain a polygonal approximation to $C$ by dividing the interval $[0; \frac{1}{4}]$ into
8.3. ARC LENGTH

Let \( n \) subintervals with dividing points \( t_0, t_1, \ldots, t_n \). The point \( P_i(x(t_i); y(t_i)) \) lies on \( C \) and the polygon with vertices \( A = P_0; P_1; \ldots; P_n = B \) is an approximation to \( C \). The length \( jP_{i-1}P_i \) of the chord connecting \( P_{i-1} \) and \( P_i \) is

\[
|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}
\]

The length \( L \) of \( C \) is approximately the length of this polygon and the approximation gets better and better as \( n \) increases. We define the length of \( C \) with equation (8.3); as the limit of the lengths of these inscribed polygons (if the limit exists)

\[
L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|
\tag{8.4}
\]

We call the curve \( C \) rectifiable.

The length of the inscribed polygon is

\[
L = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}
\]

By applying the Mean Value Theorem to \( x \) and \( y \); we find there exist \( \xi_i \) in \([t_{i-1}, t_i]\) such that

\[
\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\left[x(\xi_i)\right]^2 + \left[y(\xi_i)\right]^2} \cdot \xi_i
\]

where \( \xi_i = t_i - t_{i-1} \). Let \( \Delta = \max_{i \geq 1} \xi_i \); By Definition (8.4),

\[
L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\left[x(\xi_i)\right]^2 + \left[y(\xi_i)\right]^2} \cdot \xi_i
\]

where \( \xi_i = t_i - t_{i-1} \). An estimate for \( \xi_i \):

\[
j \cdot \xi_i = \frac{\sqrt{\left[x(\xi_i)\right]^2 + \left[y(\xi_i)\right]^2}}{\sqrt{\left[x(\xi_i)\right]^2 + \left[y(\xi_i)\right]^2} \cdot \xi_i}
\]

\[
= \frac{\sqrt{\left[x(\xi_i)\right]^2 + \left[y(\xi_i)\right]^2}}{\sqrt{\left[x(\xi_i)\right]^2 + \left[y(\xi_i)\right]^2} \cdot \xi_i}
\]

\[
= \frac{\sqrt{\left[x(\xi_i)\right]^2 + \left[y(\xi_i)\right]^2}}{\sqrt{\left[x(\xi_i)\right]^2 + \left[y(\xi_i)\right]^2} \cdot \xi_i}
\]
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\[ \mathbf{6} \]

\[ j_0 (\mathbf{y}) + j_0 (\mathbf{y}) j_0 (\mathbf{y}) \frac{\mathbf{x} + [\mathbf{y}] + [\mathbf{y}] + [\mathbf{y}]}{\mathbf{x} + [\mathbf{y}] + [\mathbf{y}] + [\mathbf{y}]} \cdot j_0 (\mathbf{y}) j_0 (\mathbf{y}) \]

Since \( y_0 (t) \) is continuous on \([a; b]\), it's uniformly continuous on \([a; b]\). 8'' > 0; 9± > 0; as \( \alpha = \max_{1 \cdot i \cdot n \cdot t \cdot t} \leq \pm \)

we have

\[ j_0 (\mathbf{y}) j_0 (\mathbf{y}) \]

since both \( \mathbf{x} \) and \( \mathbf{y} \) are in \([a; b]\); so \( \lim_{0 \to 1} \alpha \)

\[ L = \lim_{0 \to 1} \alpha \]

We recognize this expression as being equal to

\[ L = \int_0^1 \mathbf{x}^2 + [\mathbf{y}]^2 \]

since \( x_0 (t) \) and \( y_0 (t) \) are continuous and in turn \( [\mathbf{x}]^2 + [\mathbf{y}]^2 \) is integrable.

Remark 1 \( \lim_{0 \to 1} \alpha \)

2 If the curve \( C \) is given by the equation \( y = f(x); a \cdot x \cdot b \) and \( f \) is continuous on \([a; b]\), the arc length of \( C \) is

\[ L = \int_a^b \frac{\mathbf{x}^2 + [\mathbf{y}]^2}{1 + [\mathbf{y}]} \]

2 If the curve \( C \) is given by the polar equation \( r = r(\mu); a \cdot \mu \cdot b \) and \( r \) is continuous on \([a; b]\); then we may think of the curve \( C \) as being parametrized by the equations

\[ x = r(\mu) \cos \mu \]
\[ y = r(\mu) \sin \mu \]

Then the arc length of \( C \) is

\[ L = \int_a^b \frac{\mathbf{r}(\mu)^2 + [\mathbf{y}]^2}{\mathbf{r}(\mu)^2 + [\mathbf{y}]^2} \]

\[ = \int_a^b \frac{\mathbf{r}(\mu)^2 + [\mathbf{y}]^2}{\mathbf{r}(\mu)^2 + [\mathbf{y}]^2} \]
For the space curve $C$ given by the equations $x = x(t); y = y(t); z = z(t)$; the arc length of $C$ is

$$L = \int_{1}^{2} \sqrt{\left[\frac{dx(t)}{dt}\right]^2 + \left[\frac{dy(t)}{dt}\right]^2 + \left[\frac{dz(t)}{dt}\right]^2} dt$$