Chapter 8

The application and the approximate computation of definite integral

8.5 Center of mass (Centroid)

8.5.1 Center of mass for a curve

The center of mass is the point of a plate where the plate balances horizontally. We first consider a simpler case where two masses \( m_1 \) and \( m_2 \) are attached to the ends of a rod of negligible mass with the fulcrum at the distances \( d_1 \) and \( d_2 \) to the ends of the rod. The rod will balance if

\[
m_1 d_1 = m_2 d_2
\]

It's called the Law of the Lever, discovered by Archimedes by experiment.

Now suppose that the rod lies on the x-axis with \( m_1 \) at \( x_1 \), \( m_2 \) at \( x_2 \) and the center of mass at \( \bar{x} \). Then

\[
m_1 (\bar{x} - x_1) = m_2 (x_2 - \bar{x})
\]

\[
\Rightarrow \quad \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}
\]

(8.1)

\( m_1 x_1 \) and \( m_2 x_2 \) are called the moments of the masses \( m_1 \) and \( m_2 \) (with respect to the origin).
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Generally, if we have a system of \( n \) masses \( m_1, m_2, \ldots, m_n \) located at \( x_1, x_2, \ldots, x_n \) on the \( x \)-axis respectively, then the center of mass of the system is

\[
\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}
\]  

(8.2)

Actually suppose that (8.2) holds for \( n - 1 \). Then the center of mass for \( m_1, m_2, \ldots, m_{n-1} \) is

\[
\bar{x}' = \frac{\sum_{i=1}^{n-1} m_i x_i}{\sum_{i=1}^{n-1} m_i}
\]

Then by (8.1)

\[
\bar{x} = \frac{\left(\sum_{i=1}^{n-1} m_i\right) \left(\sum_{i=1}^{n-1} m_i x_i\right) + m_n x_n}{m_n + \sum_{i=1}^{n-1} m_i}
\]

\[
= \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}
\]

Done.

Now we consider a system of \( n \) particles with masses \( m_1, m_2, \ldots, m_n \) located at \( (x_1, y_1), \ldots, (x_n, y_n) \) in the \( xy \)-plane. Similarly as in the one dimensional case, the coordinates of the center of mass are given by

\[
\bar{x} = \frac{M_y}{m} \quad \bar{y} = \frac{M_x}{m}
\]

where \( m = \sum_{i=1}^{n} m_i \) is the total mass of the system and

\[
M_x = \sum_{i=1}^{n} m_i x_i \quad M_y = \sum_{i=1}^{n} m_i y_i
\]

are the moments of the system about the \( x \)-axis and \( y \)-axis respectively.

In the following we consider a smooth curve \( l \) in the plane. Divide \( l \) into \( n \) segments \( \Delta l_1, \Delta l_2, \ldots, \Delta l_n \) with the length \( \Delta s_1, \Delta s_2, \ldots, \Delta s_n \). The mass of \( \Delta l_i \) is \( \rho \Delta s_i \), where \( \rho \) is the density of the curve. Choose a sample point \( (\xi_i, \eta_i) \) in \( \Delta l_i \) and think of the segment as the mass point with the mass of the segment concentrated at \( (\xi_i, \eta_i) \). Then the coordinates of the center of mass of \( l \) are approximately given by

\[
\bar{x} \approx \frac{\sum_{i=1}^{n} \xi_i \rho \Delta s_i}{\sum_{i=1}^{n} \rho \Delta s_i} = \frac{\sum_{i=1}^{n} \xi_i \Delta s_i}{\sum_{i=1}^{n} \Delta s_i}
\]

\[
\bar{y} \approx \frac{\sum_{i=1}^{n} \eta_i \rho \Delta s_i}{\sum_{i=1}^{n} \rho \Delta s_i} = \frac{\sum_{i=1}^{n} \eta_i \Delta s_i}{\sum_{i=1}^{n} \Delta s_i}
\]
Let $\lambda = \max\{\Delta s_i\} \to 0$ we get the coordinates of the center of mass of $l$:

$$
\bar{x} = \frac{\int_l x \, ds}{\int_l ds}, \quad \bar{y} = \frac{\int_l y \, ds}{\int_l ds}
$$

If the density $\mu$ is not constant but a function of $x$, then in a similar way we can get the coordinates of the center of mass of $l$:

$$
\bar{x} = \frac{\int_l x \mu(x) \, ds}{\int_l \mu(x) \, ds} = \frac{\int_l x \, dm}{m},
$$

$$
\bar{y} = \frac{\int_l y \mu(x) \, ds}{\int_l \mu(x) \, ds} = \frac{\int_l y \, dm}{m}.
$$

**Example 1** Find the center of mass of the arc of semi circle, which is given by

$$
y = \sqrt{r^2 - x^2}
$$

**Solution:** By the Symmetry Principle $\bar{x} = 0$. And

$$
\bar{y} = \frac{\int_{-r}^{r} y \sqrt{1 + y'^2} \, dx}{\int_{-r}^{r} \sqrt{1 + y'^2} \, dx}
$$

$$
= \frac{2\pi \int_{-r}^{r} y \sqrt{1 + y'^2} \, dx}{2\pi r}
$$

$$
= \frac{4\pi r^2}{2\pi r} = \frac{2r}{\pi}
$$

where $2\pi \int_{-r}^{r} y \sqrt{1 + y'^2} \, dx$ is the area of the surface of revolution obtained by rotating the curve $y = \sqrt{r^2 - x^2}$ about the $x$-axis, i.e. the area of the surface of the sphere of the radius $r$.

### 8.5.2 Center of mass for a flat plate

We consider a flat plate with uniform density $\rho$ that occupies the region $\mathcal{R}$ of the plane. First by the symmetry principle the centroid of a rectangle is its center. Moments should be defined so that the entire mass of the region is concentrated at the centroid, then the moment remains unchanged. Also, the moment of the union of the two nonoverlapping regions should be the sum of the moments of the individual regions.
Suppose that the region $\mathcal{R}$ is bounded by the graph of $f \geq 0$, $x = a, x = b$ and $x$-axis. Divide the interval $[a, b]$ into $n$ equal subintervals by the points

$$a = x_0 < x_1 < \ldots < x_n = b$$

and choose the $x_i^*$ to be the midpoint of the $i$-th subinterval $[x_{i-1}, x_i]$. The centroid of the $i$-th approximating rectangle $R_i$ is its center $C_i(x_i^*, f(x_i^*))$. Its mass is

$$\rho f(x_i^*) \Delta x_i$$

The moment of $R_i$ about $y$-axis is

$$M_y(R_i) = (\rho f(x_i^*) \Delta x_i) x_i^*$$

Adding these moments, we obtain the moment of the polygonal approximation to $\mathcal{R}$ and then by taking the limit as $\|P\| \to 0$ we get the moment of the region $\mathcal{R}$ about $y$-axis

$$M_y = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \rho x_i^* f(x_i^*) \Delta x_i = \rho \int_{a}^{b} x f(x) dx$$

In a similar fashion we compute the moment of the moment of $R_i$ about $x$-axis as the product of its mass and the distance from $C_i$ to $x$-axis:

$$M_x(R_i) = (\rho f(x_i^*) \Delta x_i) \frac{1}{2} f(x_i^*)$$

$$= \frac{1}{2} \rho \left| f(x_i^*) \right|^2 \Delta x_i$$

So adding these moments and taking the limit we get the moment of the region $\mathcal{R}$ about $x$-axis

$$M_x = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \frac{1}{2} \rho \left| f(x_i^*) \right|^2 \Delta x_i = \frac{\rho}{2} \int_{a}^{b} \left| f(x) \right|^2 dx$$

So the center of mass is

$$\overline{x} = \frac{M_y}{m} = \frac{\rho \int_{a}^{b} x f(x) dx}{\rho \int_{a}^{b} f(x) dx} = \frac{\int_{a}^{b} x f(x) dx}{\int_{a}^{b} f(x) dx}$$

$$\overline{y} = \frac{M_x}{m} = \frac{\frac{\rho}{2} \int_{a}^{b} \left| f(x) \right|^2 dx}{\rho \int_{a}^{b} f(x) dx} = \frac{\int_{a}^{b} \frac{1}{2} \left| f(x) \right|^2 dx}{\int_{a}^{b} f(x) dx}$$
If the region \( R \) is bounded by two curves \( y = f(x) \) and \( y = g(x) \), where \( f(x) \geq g(x) \), and the area of \( R \) is \( A \), then the center of mass is

\[
\bar{x} = \frac{1}{A} \int_a^b x [f(x) - g(x)] \, dx
\]
\[
\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} \left\{ (f(x))^2 - (g(x))^2 \right\} \, dx
\]

**Example 2** Find the center of mass of a semicircular plate of radius \( r \).

*Solution:* Let the semicircle be defined by \( y = \sqrt{r^2 - x^2} \) and \( a = -r, b = r \). By the symmetry principle \( \bar{x} = 0 \). The area of the region is \( A = \pi r^2 / 2 \). So

\[
\bar{y} = \frac{1}{A} \int_{-r}^r \frac{1}{2} \left( \sqrt{r^2 - x^2} \right)^2 \, dx
\]
\[
= \frac{1}{\pi r^2 / 2} \int_{-r}^r \frac{1}{2} \left( \sqrt{r^2 - x^2} \right)^2 \, dx
\]
\[
= \frac{1}{\pi r^2 / 2} \int_{-r}^r \frac{1}{2} \left( r^2 - x^2 \right) \, dx
\]
\[
= \frac{4r}{3\pi}
\]

The center of mass is \((0, \frac{4r}{3\pi})\).

**Example 3** Find the center of mass of the region bounded \( y = x \) and the parabola \( y = x^2 \).

*Solution:* We take \( f(x) = x \) and \( g(x) = x^2 \), \( a = 0 \), and \( b = 1 \). So

\[
A = \int_0^1 (x - x^2) \, dx
\]
\[
= \frac{1}{6}
\]

Therefore

\[
\bar{x} = \frac{1}{A} \int_a^b x [f(x) - g(x)] \, dx
\]
\[
= \frac{1}{3} \int_0^1 x(x - x^2) \, dx = \frac{1}{2}
\]
\[
\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} \left\{ (f(x))^2 - (g(x))^2 \right\} \, dx
\]
\[
= \frac{1}{6} \int_0^1 \frac{1}{2} (x^2 - x^4) \, dx = \frac{2}{5}
\]
The centroid is \(\left(\frac{1}{2}, \frac{1}{2}\right)\).

Exercises: Page 325, Ch8 5-1, 5-2.