Chapter 9

Series

9.4 Series with any terms

Definition 1 A series $\sum u_i$ is called absolutely convergent if the series of the absolute values $\sum |a_i|$ is convergent.

Definition 2 A series $\sum u_i$ is called conditionally convergent if it is convergent but not absolutely convergent.

Example 3 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 4 If a series is absolutely convergent, then it is conditionally convergent.

Proof: Way 1:
Suppose the series $\sum u_i$ is absolutely convergent. By Cauchy Convergence Theorem, $\forall \varepsilon > 0, \exists N > 0$ such that $\forall p > 0$ we have

$$|u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}| < \varepsilon$$

whenever $n > N$. So

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < |u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}| < \varepsilon$$

Then by Cauchy Convergence Theorem the series $\sum u_i$ is convergent.

Way 2:

$$0 \leq u_n + |v_n| \leq 2|v_n|$$
That the series \( \sum |u_i| \) converges implies \( \sum u_n + |u_n| \) is convergent. So
\[
\sum u_n = \sum u_n + |u_n| - \sum |u_i|
\]
implies that \( \sum u_i \) is convergent.

**Example 5** Test the convergence of the series \( \sum_{n=1}^{\infty} (-1)^{n+1} x^n \).

### 9.4.1 Alternating series

**Definition 6** An alternating series is a series whose terms are alternately positive and negative.

**Theorem 7** The Alternating Series Test (Leibniz) If the alternating series
\[
\sum_{n=1}^{\infty} (-1)^{n+1} u_n
\]
satisfies

(i) \( u_{n+1} \leq u_n \) for all \( n \)

(ii) \( \lim_{n \to \infty} u_n = 0 \)

then the series is convergent. Let \( R_n = \sum_{i=n+1}^{\infty} (-1)^i u_i \) be the remainder of \( \sum_{n=1}^{\infty} (-1)^n u_n \). Then
\[
|R_n| \leq u_{n+1}
\]

**Proof:** Let \( S_n \) be the \( n \)-th partial sum of the series \( \sum (-1)^n u_n \). Then we consider the even partial sums:

\[
S_{2n} = (u_2 - u_2) + (u_3 - u_4) + \cdots + (u_{2n-1} - u_{2n})
\]

\[
S_{2n+2} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2n-1} - u_{2n}) + (u_{2n+1} - u_{2n+2})
\]

By \( u_{n+1} \leq u_n \), \( S_{2n+2} = S_{2n} + (u_{2n+2} - u_{2n+1}) \geq S_{2n} \). And
\[
S_{2n} = u_1 - (u_2 - u_3) - \cdots - (u_{2n-2} - u_{2n-1}) - u_{2n} \leq u_1
\]
since every term in brackets is nonnegative. So \( \{ S_{2n} \} \) is increasing and bounded above. Therefore the limit \( \lim_{n \to \infty} S_{2n} \) exists. For the odd partial sum we have

\[
\lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} (S_{2n} + u_{2n+1}) = \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} u_{2n+1} = \lim_{n \to \infty} S_{2n} + 0 = \lim_{n \to \infty} S_{2n}
\]

Since both the even and odd partial sums converge to the same limit, so does \( \sum (-1)^n u_n \). By

\[
S_{2n} = u_1 - (u_2 - u_3) - \cdots - (u_{2n-2} - u_{2n-1}) - u_{2n}
\]

and

\[
S_{2n+1} = u_1 - (u_2 - u_3) - \cdots - (u_{2n-2} - u_{2n-1}) - (u_{2n} - u_{2n})
\]

and \( u_i - u_{i+1} \geq 0, i = 2, 3, \ldots \), we know

\[
0 \leq \sum_{n=1}^{\infty} (-1)^{n+1} u_n \leq u_1
\]

Multiplying the above expression by \(-1\) we get

\[
-u_1 \leq \sum_{n=1}^{\infty} (-1)^n u_n \leq 0
\]

So a Leibniz series has the same sign as its first term \( u_1 \) and its absolute value is less than or equal to \( u_1 \). \( R_n = \sum_{i=n+1}^{\infty} (-1)^i u_i \) is also a Leibniz series. So \( |R_n| \leq u_{n+1} \).

**Example 8** \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^s} \) \( (s > 0) \).

**Example 9** \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \) \( (s > 0) \).
9.4.2 Abel Test and Dirichlet Test

**Abel transformation:** For the sum

\[ S = \sum_{i=1}^{m} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_m b_m \]

Abel Niels gave an elementary transformation. Let

\[ B_n = \sum_{i=1}^{n} b_i, \quad i = 1, 2, \ldots, m \]

Then

\[ b_1 = E_1, \quad b_n = E_n - E_{n-1}, \quad i = 1, 2, \ldots, m \]

And

\[ S = \sum_{i=1}^{m} a_i b_i \]

\[ = a_1 E_1 + a_2 (E_2 - E_1) + a_3 (E_3 - E_2) + \cdots + a_m (E_m - E_{m-1}) \]

\[ = (a_1 - a_2) E_1 + (a_2 - a_3) E_2 + \cdots + (a_{m-1} - a_m) E_{m-1} + a_mE_m \]

\[ = \sum_{i=1}^{m-1} (a_i - a_{i+1}) E_i + a_mE_m \]

\[ = a_mE_m - \sum_{i=1}^{m-1} (a_{i+1} - a_i) E_i \]

Compare the last expression to the formula for the integration by parts:

\[ \int_a^b f(x) g(x) \, dx = f(x) G(x) \bigg|_{a}^{b} - \int_a^b G(x) \, df(x) \]

\[ = f(b) G(b) - \int_a^b G(x) \, df(x) \]

where \( G(x) = \int_a^x g(x) \, dx \). \( E_i \) in the Abelian transformation corresponds to \( G(x) \) and \( (a_{i+1} - a_i) \) to \( df(x) \).

**Lemma 10** (Abel) If

(i) \( \{a_i\} \ (i = 1, 2, \ldots, m) \) is monotonous;

(ii) \( \{B_i\} \ (i = 1, 2, \ldots, m) \) is bounded, \( |B_i| \leq M \).
Then \(|S| = |\sum_{i=1}^{m} a_i b_i| \leq M (|a_1| + 2 |a_m|)\)

Proof:

\[
|S| = \left| a_m B_m - \sum_{i=1}^{m-1} (a_{i+1} - a_i) E_i \right|
\leq \sum_{i=1}^{m-1} (a_{i+1} - a_i) E_i + |a_m B_m|
\leq M \sum_{i=1}^{m-1} |(a_{i+1} - a_i)| + M |a_m|
= M \left| \sum_{i=1}^{m-1} a_{i+1} - a_i \right| + M |a_m|
= M |a_m - a_1| + M |a_m|
\leq M (|a_1| + 2 |a_m|)
\]

where \(\sum_{i=1}^{m-1} |(a_{i+1} - a_i)| = \sum_{i=1}^{m-1} a_{i+1} - a_i\) because each \((a_{i+1} - a_i)\) has the same sign.

Corollary 11 If \(a_i \geq 0\) \((i = 1, 2, \ldots, m)\) and \(a_1 \geq a_2 \geq \cdots \geq a_m\) then \(|S| \leq M a_1\)

\[
|S| \leq M \sum_{i=1}^{m-1} |(a_{i+1} - a_i)| + M |a_m|
\leq M (a_1 - a_m) + M a_m
\leq M a_1
\]

Abel Test: If (i) the series \(\sum_{n=1}^{\infty} b_n\) converges; (ii) the sequence \(\{a_n\}\) is monotonous and bounded, \(|a_n| \leq K\) \((i = 1, 2, \ldots, m)\). Then the series \(\sum_{i=1}^{\infty} a_i b_i\) converges.

Proof: Estimate the sum

\[
\sum_{k=n+1}^{n+m} a_k b_k = \sum_{i=1}^{m} a_{i+n} b_{i+n}
\]

By (i) \(\sum_{n=1}^{\infty} b_n\) converges. So by Cauchy Convergence Theorem, \(\forall \varepsilon > 0, \exists N > 0\) such that \(\forall n > 0\) we have

\[|b_{n+1} + b_{n+2} + \cdots + b_{n+p}| < \varepsilon\]
whenever $n > N$. Apply Abel Lemma to $\{B_k\}$, $B_k = \sum_{i=1}^k b_{i+n}$ and $\{a_i\}$ with $M = \varepsilon$ and we get
\[
\left| \sum_{k=n+1}^{n+m} a_k b_k \right| \leq \varepsilon (|a_1| + 2 |a_m|) \\
\leq 3\varepsilon K
\]

So $\sum_{i=1}^\infty a_i b_i$ converges.

**Dirichlet Test** If (i) the partial sum $B_i$ of the series $\sum_{n=1}^\infty b_n$ is bounded, $|B_i| \leq M$ ($i = 1, 2, \ldots, m$); (ii) the sequence $\{a_n\}$ approaches zero monotonously. Then the series $\sum_{i=1}^\infty a_i b_i$ converges.

Proof: By $a_n \to 0$ we have $\forall \varepsilon > 0, \exists N > 0$ we have $|a_n| < \varepsilon$ whenever $n > N$. By (i) we have
\[
|b_{n+1} + b_{n+2} + \cdots + b_{n+m}| = |B_{n+m} - B_n| \leq 2M
\]

whenever $n > N$. For any $m > 0$ we have
\[
\left| \sum_{k=n+1}^{n+m} a_k b_k \right| \leq 2M (|a_{n+1}| + 2 |a_{n+m}|) < 6M \varepsilon
\]

So $\sum_{i=1}^\infty a_i b_i$ converges.

We can obtain Abel Test from Dirichlet Test. The Leibniz Theorem on alternating series is an corollary of Dirichlet Test.

Exercises: (II) Page 29, 4-1(2), 4-1(3), 4-1(6), 4-1(7), 4-2, 4-3(3), 4-3(4), 4-4.