Chapter 9

Series

9.4 Series with any terms

9.4.1 Abel Test and Dirichlet Test

Lemma 1 (Abel) If

(i) \{a_i\} (i = 1, 2, \ldots, m) is monotonous;
(ii) \{E_i\} (i = 1, 2, \ldots, m) is bounded, |E_i| \leq M.

Then |S| = \left| \sum_{i=1}^{m} a_i b_i \right| \leq M (|a_1| + 2 |a_m|)

Proof:

\[
|S| = \left| b_mE_m - \sum_{i=1}^{m-1} (a_{i+1} - a_i)E_i \right|
\leq \sum_{i=1}^{m-1} |(a_{i+1} - a_i)E_i| + |a_mE_m|
\leq M \sum_{i=1}^{m-1} |(a_{i+1} - a_i)| + M |a_m|
= M \sum_{i=1}^{m-1} |a_{i+1} - a_i| + M |a_m|
= M |a_m - a_1| + M |a_m|
\leq M (|a_1| + 2 |a_m|)
\]

where \sum_{i=1}^{m-1} |(a_{i+1} - a_i)| = \sum_{i=1}^{m-1} |a_{i+1} - a_i| because each \(a_{i+1} - a_i\) has the same sign.
Corollary 2 If \( a_i \geq 0 \) \((i = 1, 2, \ldots, m)\) and \( a_1 \geq a_2 \geq \cdots \geq a_m \) then 
\[
|S| \leq Ma_1
\]

\[
|S| \leq M \sum_{i=1}^{m-1} |a_{i+1} - a_i| + M |a_m|
\]

\[
\leq M (a_1 - a_m) + Ma_m
\]

\[
\leq Ma_1
\]

**Abel Test** If (i) the series \( \sum_{n=1}^\infty b_n \) converges; (ii) the sequence \( \{a_n\} \) is monotonous and bounded, \( |a_n| \leq K \) \((i = 1, 2, \ldots, m)\). Then the series \( \sum_{i=1}^\infty a_i b_i \) converges.

**Proof:** Estimate the sum

\[
\sum_{k=n+1}^{n+m} a_k b_k = \sum_{i=1}^m a_{i+n} b_{i+n}
\]

By (i) \( \sum_{n=1}^\infty b_n \) converges. So by Cauchy Convergence Theorem, \( \forall \epsilon > 0, \exists N > 0 \) such that \( \forall p > 0 \) we have

\[
|b_{n+1} + b_{n+2} + \cdots + b_{n+p}| < \epsilon
\]

whenever \( n > N \). Apply Abel Lemma to \( \{B_k\}, B_k = \sum_{i=1}^k b_{i+n} \) and \( \{a_i\} \) with \( M = \epsilon \) and we get

\[
\left| \sum_{k=n+1}^{n+m} a_k b_k \right| \leq \epsilon (|a_1| + 2 |a_m|)
\]

\[
\leq 3\epsilon K
\]

So \( \sum_{i=1}^\infty a_i b_i \) converges.

**Dirichlet Test** If (i) the partial sum \( B_i \) of the series \( \sum_{n=1}^\infty b_n \) is bounded, \( |B_i| \leq M \) \((i = 1, 2, \ldots, m)\); (ii) the sequence \( \{a_n\} \) approaches zero monotonously. Then the series \( \sum_{i=1}^\infty a_i b_i \) converges.

**Proof:** By \( a_n \to 0 \) we have \( \forall \epsilon > 0, \exists N > 0 \) we have \( |a_n| < \epsilon \) whenever \( n > N \). By (i) we have

\[
|b_{n+1} + b_{n+2} + \cdots + b_{n+p}| = |B_{n+p} - B_n| \leq 2M
\]
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whenever \( n > N \). For any \( m > 0 \) we have

\[
\left| \sum_{k=n+1}^{n+m} a_k b_k \right| \leq 2M \left( |a_{n+1}| + 2|a_{n+m}| \right) < 6M \varepsilon
\]

So \( \sum_{i=1}^{\infty} a_i b_i \) converges.

We can obtain Abel Test from Dirichlet Test. The Leibniz Theorem on alternating series is a corollary of Dirichlet Test.

Problem solution:

Page 5, #6. Suppose \( \lim_{n \to \infty} a_n = a < b \). Show that there must exist an \( N > 0 \) such that \( a_n < b \) whenever \( n > N \). What if \( \lim_{n \to \infty} a_n = a < b \)?

Proof: Let \( \beta_n = \sup \{ a_k \} \). Let \( \varepsilon = b - a \). Then \( \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} a_n = a \Rightarrow \exists N > 0 \) such that we have \( \beta_{n-1} < a + \varepsilon = b \) whenever \( n > N \). Therefore \( a_n \leq \beta_{n-1} < b \) whenever \( n > N \).

If \( \lim_{n \to \infty} a_n = a < b \), by Theorem 2 on page 3 there are infinitely many terms \( a_n < a + \varepsilon = b \).

Page 12, #3. Let \( \sum a_n \) be a series with positive terms. If the series obtained by adding brackets to some terms in \( \sum a_n \) converge, then \( \sum a_n \) converge.

Proof: Suppose that the new series is

\[
(a_1 + \cdots + a_{i_1}) + \cdots + (a_{i_{n-1}+1} + \cdots + a_{i_n}) + \cdots
\]

Let \( S_n \) and \( S_n' \) be the \( n \)-th partial sums of the series \( \sum a_n \) and the new series. Then

\[
S_n' = S_i_n
\]

Both \( \{ S_n \} \) and \( \{ S_n' \} \) are increasing sequences. \( \lim S_n' = \lim S_n \). So the increasing sequence \( \{ S_n \} \) has a convergent subsequence and then \( S_n \) must converge.

Page 12, #4. Determine the \( x \) range of convergence of the following series:

1. \( \sum \frac{1}{(1+x)^n} \)
2. \( \sum (\ln x)^n \)

Solution: They are geometric series. (1): \( \sum \frac{1}{(1+x)^n} \) converges iff \( \left| \frac{1}{1+x} \right| < 1 \iff x > 0 \) or \( x < -2 \)

(2): \( \sum (\ln x)^n \) converges iff \( |\ln x| < 1 \iff e^{-1} < x < e \)

Exercises: (II) Page 29, 4-3(3), 4-3(4), 4-4.