Chapter 9

Series

9.5 The properties of the absolutely convergent and conditionally convergent series

Theorem 1 Given a series \( \sum_{n=1}^{\infty} u_n \) we define

\[
    v_n = \frac{|u_n| + u_n}{2} = \begin{cases} u_n, & \text{if } u_n > 0 \\ 0, & \text{if } u_n \leq 0 \end{cases}
\]

\[
    w_n = \frac{|u_n| - u_n}{2} = \begin{cases} -u_n, & \text{if } u_n > 0 \\ 0, & \text{if } u_n \leq 0 \end{cases}
\]

then (1): if the series \( \sum_{n=1}^{\infty} u_n \) is absolutely convergent, then the series \( \sum_{n=1}^{\infty} v_n \) and \( \sum_{n=1}^{\infty} w_n \) are convergent.

(2): if the series \( \sum_{n=1}^{\infty} u_n \) is conditionally convergent, then the series \( \sum_{n=1}^{\infty} v_n \) and \( \sum_{n=1}^{\infty} w_n \) are divergent.

Proof: (1): First \( v_n \geq 0, w_n \geq 0 \). Then by \( v_n \leq |u_n| \) and \( w_n \leq |u_n| \) and the comparison test we know that the convergence of the series \( \sum_{n=1}^{\infty} u_n \) and \( \sum_{n=1}^{\infty} w_n \) follows from the absolute convergence of the series \( \sum_{n=1}^{\infty} u_n \).

(2): Suppose \( \sum_{n=1}^{\infty} u_n \) is convergent. Then \( w_n = \frac{|u_n| - u_n}{2} \Rightarrow |u_n| = 2w_n + v_n \) and it follows that \( \sum_{n=1}^{\infty} w_n \) is absolutely convergent. Contradiction.

Similarly we can prove it for \( \sum_{n=1}^{\infty} v_n \).

9.5.1 Rearrangement of a series

The question of whether a series is absolutely convergent or conditionally convergent has a bearing on the question of whether infinite sums behave
like finite sums.

We know that the commutativity holds for the finite sums. But how about the infinite sums?

By a rearrangement of an infinite series \( \sum_{n=1}^{\infty} u_n \), we mean a series obtained by simply changing the order of the terms.

**Theorem 2** An rearrangement \( \sum_{n=1}^{\infty} u'_n \) of an absolutely convergent series \( \sum_{n=1}^{\infty} u_n \) is still absolutely convergent.

**Proof:** (i) First assume \( \sum_{n=1}^{\infty} u_n \) is a positive absolutely convergent series. Consider the partial sum \( S'_k \) of the rearrangement \( \sum_{n=1}^{\infty} u'_n \). We have

\[
 u'_1 = u_{n_1}, \quad u'_2 = u_{n_2}, \ldots, \quad u'_k = u_{n_k}.
\]

Let \( n = \max\{n_1, n_2, \ldots, n_k\} \), then

\[
 S'_k = u'_1 + u'_2 + \cdots + u'_k \leq u_1 + u_2 + \cdots + u_n = S_n
\]

Let \( S = \sum_{n=1}^{\infty} u_n \). Since \( \lim S_n = S \) and \( S'_k \) is increasing, \( \lim S'_k \) exists and \( \sum_{n=1}^{\infty} u'_n \) is absolutely convergent. Let \( \lim S'_k = S' \). Then

\[
 S' \leq S
\]

On the other hand, \( \sum_{n=1}^{\infty} u_n \) is also a rearrangement of \( \sum_{n=1}^{\infty} u'_n \), so \( S \leq S' \). Hence \( S = S' \).

(ii) Now consider the case where \( \sum_{n=1}^{\infty} u_n \) is an arbitrary absolutely convergent series. Let \( \sum_{n=1}^{\infty} v_n \) and \( \sum_{n=1}^{\infty} w_n \) be as in Theorem 1. By Theorem 1 \( \sum_{n=1}^{\infty} v_n \) and \( \sum_{n=1}^{\infty} w_n \) are both convergent. Let \( \sum_{n=1}^{\infty} v_n = V \) and \( \sum_{n=1}^{\infty} w_n = W \). Then

\[
 \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (v_n - w_n) = V - W
\]

\[
 \sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} (v_n + w_n) = V + W
\]

For the rearrangement \( \sum_{n=1}^{\infty} |u'_n| \) of \( \sum_{n=1}^{\infty} |u_n| \), by (i) we have

\[
 \sum_{n=1}^{\infty} |u'_n| = \sum_{n=1}^{\infty} |u_n| = V + W
\]
9.5. THE PROPERTIES OF THE ABSOLUTELY CONVERGENT AND CONDITIONALLY CONVERGENT SERIES

So \( \sum_{n=1}^{\infty} w_n' \) is absolutely convergent. Let \( \sum_{n=1}^{\infty} v_n' \) and \( \sum_{n=1}^{\infty} w_n' \) be rearrangements of \( \sum_{n=1}^{\infty} v_n \) and \( \sum_{n=1}^{\infty} w_n \). Then by (i) we have

\[
\sum_{n=1}^{\infty} v_n' = \sum_{n=1}^{\infty} v_n = V, \quad \sum_{n=1}^{\infty} w_n' = \sum_{n=1}^{\infty} w_n = W
\]

By

\[
v_n' = v_n' - w_n'
\]

we get

\[
\sum_{n=1}^{\infty} (v_n' - w_n') = \sum_{n=1}^{\infty} v_n' - \sum_{n=1}^{\infty} w_n' = V - W = \sum_{n=1}^{\infty} u_n
\]

Question: If any rearrangement \( \sum_{n=1}^{\infty} u_n' \) of a convergent series \( \sum_{n=1}^{\infty} u_n \) is convergent and \( \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} u_n' \). Is it necessarily that \( \sum_{n=1}^{\infty} u_n \) is absolutely convergent?

**Example 3**

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots = \ln 2 = S
\]

(Riemann Theorem): If \( \sum_{n=1}^{\infty} u_n \) is conditionally convergent, then for any number \( r \) (including \( \infty \)) there is a rearrangement of \( \sum_{n=1}^{\infty} u_n \) that has a sum equal to \( r \).

Sketch of proof: Let

\[
u_n^+ = \frac{|u_n|} {2}, \quad u_n^- = \frac{u_n - |u_n|} {2}
\]

Take enough positive terms \( \sum u_n^+ \) so that their sum is greater than \( r \). Then add just enough negative terms \( \sum u_n^- \) so that the cumulative sum is less than \( r \). Continue this in this way and use that \( \lim u_n = 0 \).

9.5.2 The product of two infinite series

Given two convergent series \( \sum_{n=1}^{\infty} u_n \) and \( \sum_{n=1}^{\infty} v_n \) define the Cauchy product \( \sum_{n=1}^{\infty} c_n \) of the series \( \sum_{n=1}^{\infty} u_n \) and \( \sum_{n=1}^{\infty} v_n \) where

\[
c_n = u_1 v_n + u_2 v_{n-1} + \cdots + u_n v_1
\]

**Theorem 4 (Cauchy)** Suppose the series \( \sum_{n=1}^{\infty} u_n \) and \( \sum_{n=1}^{\infty} v_n \) are absolutely convergent and that \( \sum_{n=1}^{\infty} v_n = U \) and \( \sum_{n=1}^{\infty} v_n = V \). Then the sum of the product \( u_n v_k \) \( (k = 1, 2, 3, \ldots) \) of their terms by any order of \( u_n v_k \) is also absolutely convergent and the sum is \( UV \).
CHAPTER 9. SERIES

Proof: Let $w_1, w_2, w_3, \ldots, w_n, \ldots$ be the sequence of the product $w_kv_k$ by some order. Consider the series

$$|w_1| + |w_2| + |w_3| + \cdots + |w_n| + \cdots$$

Let $S^*_n$ be the partial sum

$$S^*_n = \sum_{k=1}^{n} |w_k| = \sum_{k=1}^{n} |u_{n_k}v_{m_k}|$$

Let

$$\nu = \max(n_1, n_2, n_3, \ldots, n_n, m_n)$$

and

$$U^*_\nu = |u_1| + |u_2| + \cdots + |u_\nu|$$

$$V^*_\nu = |v_1| + |v_2| + \cdots + |v_\nu|$$

Since $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are absolutely convergent, $U^*_\nu$ and $V^*_\nu$ are bounded. Furthermore

$$S^*_n = \sum_{k=1}^{n} |u_{n_k}v_{m_k}| \leq \left(\sum_{k=1}^{n} |u_{n_k}|\right) \left(\sum_{k=1}^{n} |v_{m_k}|\right) = U^*_\nu V^*_\nu$$

So $S^*_n$ is bounded and then $\sum_{n=1}^{\infty} w_n$ is absolutely convergent. Then any rearrangement of $\sum_{n=1}^{\infty} w_n$ is absolutely convergent and their sums are same by Theorem 2.

Consider the series obtained by square method and bracket the terms as follows:

$$\sum_{n=1}^{\infty} a_n = u_1v_1 + (u_1v_2 + u_2v_3 + u_2v_4) + (u_1v_5 + u_2v_6 + u_3v_7 + u_3v_8 + u_3v_9) + \cdots$$

By Property 3 on page 7 the bracketing does not change the convergence or the sum. Let $U_n$, $V_n$ and $A_n$ be the partial sums of $\sum_{n=1}^{\infty} u_n$, $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} a_n$. Then we have

$$A_n = U_n V_n$$

$$\Rightarrow$$

$$\lim A_n = \lim (U_n V_n) = UV$$

Q.E.D.

Exercises: Ch 9, Page 37, 5-1, 5-2.