Chapter 7

Definite Integral

7.1 The concept of the definite Integral

7.2 The condition for the existence of the definite Integral

Theorem 1 (First sufficient and necessary conditions for the existence of the definite integral) The function \( f(x) \) is integrable if and only if

\[
\lim_{\max\{\Delta_n\} \to 0} L(f, P) = \lim_{\max\{\Delta_n\} \to 0} U(f, P)
\]

Proof: proof of necessity: \( f(x) \) is integrable. So \( \forall \varepsilon > 0, \exists \delta > 0 \) such that \( \forall \) partition \( P = \{x_0, \ldots, x_n\} \) with \( \lambda(P) < \delta \) and \( \forall \xi_i \in [x_{i-1}, x_i] \) we have

\[
\left| \sum_{i=1}^{n} f(\xi_i) \Delta x_i - I \right| < \varepsilon/2
\]

where \( I = \int_{a}^{b} f(x) \, dx \). Let \( m_i = \inf\{ f(x) : x \in [x_{i-1}, x_i] \} \), then we have \( \eta_i \in [x_{i-1}, x_i] \) such that

\[
0 \leq f(\eta_i) - m_i \leq \frac{\varepsilon}{2(b - a)}
\]

Then

\[
\left| \sum_{i=1}^{n} f(\eta_i) \Delta x_i - L(f, P) \right| = \left| \sum_{i=1}^{n} (f(\eta_i) - m_i) \Delta x_i \right| = \left| \sum_{i=1}^{n} (f(\eta_i) - f(x)) \Delta x_i \right| \quad (7.1)
\]
\[ \leq \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon/2 \]

On the other hand we have
\[ \left| \sum_{i=1}^{n} f(\eta_i) \Delta x_i - I \right| < \varepsilon/2 \quad (7.2) \]

So by (7.1) and (7.2) we get
\[ |L(f, P) - I| < \sum_{i=1}^{n} f(\eta_i) \Delta x_i - L(f, P) \bigg| + \sum_{i=1}^{n} f(\eta_i) \Delta x_i - I \bigg| < \varepsilon \]

i.e.
\[ \lim_{\max \{\Delta x_i\} \to 0} L(f, P) = I \]

Similarly we can prove that
\[ \lim_{\max \{\Delta x_i\} \to 0} U(f, P) = I \]

Hence
\[ \lim_{\max \{\Delta x_i\} \to 0} L(f, P) = \lim_{\max \{\Delta x_i\} \to 0} U(f, P) \]

Proof of sufficiency: suppose \( \lim_{\max \{\Delta x_i\} \to 0} L(f, P) = \lim_{\max \{\Delta x_i\} \to 0} U(f, P) \).
Then \( \forall \) partition \( P = \{x_0, \ldots, x_n\} \) with and \( \forall \xi_i \in [x_{i-1}, x_i] \) we have
\[ L(f, P) \leq \sum_{i=1}^{n} f(\xi_i) \Delta x_i \leq U(f, P) \]

Letting \( \lambda \to 0 \) yields
\[ \lim_{\max \{\Delta x_i\} \to 0} \sum_{i=1}^{n} f(\xi_i) \Delta x_i = I \]

Q.E.D.

**Remark 1** The function \( f(x) \) is integrable if and only if
\[ \lim_{\max \{\Delta x_i\} \to 0} (L(f, P) - U(f, P)) = 0 \]
Remark 2 \( \omega_i = M_i - m_i \) is called the amplitude of \( f \) on \([x_{i-1}, x_i]\). \( \sum_{i=1}^{n} \omega_i \Delta x_i \) represents the difference of the sums of two sets of rectangles which enclose and inscribe the curve of \( f(x) \) respectively.

Theorem 2 (Second sufficient and necessary conditions for the existence of the definite integral) The function \( f(x) \) is integrable if and only if \( \forall \varepsilon > 0 \) and \( \sigma > 0, \exists \delta > 0 \) such that \( \forall \) partition \( P = \{x_0, \ldots, x_n\} \) with \( \lambda(P) < \delta \) we have the sum \( \sum \Delta x_\nu \) of the lengths of the subintervals \( \Delta x_\nu \) for which the amplitude \( \omega_\nu \geq \varepsilon \) is less than \( \sigma \).

Proof: proof of necessity: suppose \( f(x) \) is integrable. \( \forall \varepsilon > 0 \) and \( \sigma > 0, \exists \delta > 0 \) such that \( \forall \) partition \( P = \{x_0, \ldots, x_n\} \) with \( \lambda(P) < \delta \) we have

\[
\sum_{i=1}^{n} \omega_i \Delta x_i < \sigma \varepsilon
\]

So for the amplitude \( \omega_\nu \geq \varepsilon \) we have

\[
\varepsilon \sum \Delta x_\nu \leq \sum \omega_\nu \Delta x_\nu \leq \sum_{i=1}^{n} \omega_i \Delta x_i < \sigma \varepsilon
\]

which implies

\[
\sum \Delta x_\nu \leq \sigma
\]

\( \sum \Delta x_\nu \) denotes the sum over the subintervals \( \Delta x_\nu \) for which the amplitude \( \omega_\nu \geq \varepsilon \).

Proof of sufficiency: let \( \Delta x_\nu \) be the length of the subinterval for which \( \omega_\nu < \varepsilon \). And \( \sum_{\nu} \) denotes the sum over the such subintervals \( \Delta x_\nu \). Let - be the amplitude of \( f \) over \([a, b]\). Then

\[
\sum_{i=1}^{n} \omega_i \Delta x_i = \sum_{\nu} \omega_\nu \Delta x_\nu + \sum_{\nu'} \omega_{\nu'} \Delta x_{\nu'} < - \sum_{\nu} \Delta x_\nu + \varepsilon \sum_{\nu'} \Delta x_{\nu'} < - \sigma + \varepsilon (b - a)
\]

Because \( \sigma \) and \( \varepsilon \) are arbitrary numbers \( \lim_{\lambda \to 0} \sum_{i=1}^{n} \omega_i \Delta x_i \).
7.2.1 The class of integrable functions

1. The continuous function \( f(x) \) over \([a, b]\) is integrable.
   
   Proof: \( f(x) \) is continuous over \([a, b]\) then it’s uniformly continuous over \([a, b]\). So \( \forall \varepsilon > 0, \exists \delta > 0 \) such that \( \forall \) two arbitrary points \( x', x'' \in [a, b] \) with \( |x' - x''| < \delta \) we have \( |f(x') - f(x'')| < \frac{\varepsilon}{b-a} \). Then \( \forall \) partition \( P = \{x_0, \ldots, x_n\} \) with \( \lambda(P) < \delta \) we have \( M_i - m_i = \omega_i \leq \frac{\varepsilon}{b-a} \). This implies \( U(f, P) - L(F, P) = \sum_{i=1}^{n} \omega_i \Delta x_i \leq \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \varepsilon \). Therefore \( f(x) \) is integrable. Q.E.D.

2. The piecewisely continuous function \( f(x) \) is integrable over \([a, b]\).
   
   Proof: suppose that \( f(x) \) has \( k \) discontinuities \( x'_1, x'_2, \ldots, x'_k \). So \( \forall \varepsilon > 0 \) and \( \sigma > 0, \exists \delta > 0 \) and \( \delta < \frac{\varepsilon}{2k} \) such that \( \forall \) partition \( P = \{x_0, \ldots, x_n\} \) with \( \lambda(P) < \delta \) the amplitude \( \omega_i \) of \( f \) over the subinterval which does not contain any discontinuity \( x'_i \)'s is less than \( \varepsilon \) and the sum of the lengths of the subintervals over which the amplitude \( \omega_i \) of \( f \) is greater than or equal to \( \varepsilon \) is at most \( 2k \times \delta = \sigma \). Then it’s done by Theorem 6.

3. The monotone function \( f(x) \) is integrable over \([a, b]\).
   
   Proof: suppose that \( f(x) \) is increasingly monotone.\( \forall \varepsilon > 0 \) let \( \delta = \frac{\varepsilon}{f(b) - f(a)} \).
   
   Then \( \forall \) partition \( P = \{x_0, \ldots, x_n\} \) with \( \lambda(P) < \delta \) we have \( \omega_i = f(x_i) - f(x_{i-1}) \geq 0 \) and
   
   \[
   \sum_{i=1}^{n} \omega_i \Delta x_i \leq \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \Delta x_i
   \]
   
   \[
   \leq \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \delta
   \]
   
   \[
   \leq \frac{\varepsilon}{f(b) - f(a)} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))
   \]
   
   \[
   \leq \frac{\varepsilon}{f(b) - f(a)} (f(b) - f(a))
   \]
   
   \[
   < \varepsilon
   \]

So \( f(x) \) is integrable over \([a, b]\).

the http://www.actuaryjobs.com/

http://dir.sogou.com/dirsearch.jsp?classkey=C005037