Problem. Find $y(x)$ such that $y'' - 2xy' - 2y = 0$.

(This equation is second-order linear with non-constant coefficients.)

Power Series Solution. We assume that there exists a solution to the D.E. which can be represented by a Taylor series. This is not always the case.

We write

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + a_{k+1} x^{k+1} + a_{k+2} x^{k+2} + \cdots$$

Then

$$y = \quad a_0 + a_1 x + \cdots + a_k x^k + a_{k+1} x^{k+1} + \cdots$$
$$y' = \quad a_1 + 2a_2 x + \cdots + ka_k x^{k-1} + (k+1)a_{k+1} x^k + (k+2)a_{k+2} x^{k+1} + \cdots$$
$$y'' = \quad 2a_2 + 3 \cdot 2a_3 x + \cdots + (k+2)(k+1)a_{k+2} x^k + \cdots$$
$$-2xy' = \quad -2a_1 x - \cdots - 2ka_k x^k - \cdots$$
$$-2y = \quad -2a_0 - 2a_1 x + \cdots - 2a_k x^k - \cdots$$

Substituting in the equation $y'' - 2xy' - 2y = 0$ yields

$$2a_2 - 2a_0 = 0,$$
$$6a_3 - 4a_1 = 0,$$
$$(k+2)(k+1)a_{k+2} - 2(k+1)a_{k} = 0.$$

Thus $a_{k+2} = 2a_k/(k+2)$. Substituting $r = k - 2$ yields $a_r = 2a_{r-2}/r$. Therefore, for instance

$$a_8 = \frac{2a_6}{8} = \frac{2^2 a_4}{8 \cdot 6} = \frac{2^3 a_2}{8 \cdot 6 \cdot 4} = \frac{a_2}{4 \cdot 3 \cdot 2} = \frac{a_0}{4 \cdot 3 \cdot 2},$$

and in general, for any $s \geq 2$,

$$a_{2s} = \frac{2a_{2s-2}}{2s} = \frac{4a_{2s-4}}{4s(s-1)} = \frac{8a_{2s-6}}{8s(s-1)(s-2)} = \cdots = \frac{2^s a_0}{2^s s!} = \frac{a_0}{s!},$$

and for $s \geq 1$,

$$a_{2s+1} = \frac{2a_{2s-1}}{2s+1} = \frac{2^2 a_{2s-3}}{(2s+1)(2s-1)} = \cdots = \frac{2^s a_1}{(2s+1)(2s-1) \cdots 3}.$$

Thus

$$y = a_0 \sum_{s=0}^{\infty} \frac{x^{2s}}{s!} + a_1 \sum_{s=0}^{\infty} \frac{2^s x^{2s+1}}{(2s+1) \cdots 3 \cdot 1} = a_0 e^{x^2} + a_1 \sum_{s=0}^{\infty} \frac{2^s x^{2s+1}}{(2s+1) \cdots 3 \cdot 1}$$

where $a_0$ and $a_1$ are arbitrary constants.