Math 140  Lecture 6

Study Practice Exam 1 and the recommended exercises.

Functions can be added and multiplied just like numbers.

**DEFINITION.** For functions \( f, g \), define \( f+g, f-g, fg, f/g \) by

\[
(f+g)(x) = f(x) + g(x), \quad (f-g)(x) = f(x) - g(x),
\]

\[
(fg)(x) = f(x)g(x), \quad (f/g)(x) = f(x)/(g(x)).
\]

- If \( f(x) = x - 2 \), \( g(x) = 6 \), then
  \[
  (f+g)(x) = f(x) + g(x) = (x-2) + 6 = x + 4,
  \]
  \[
  (f-g)(x) = f(x) - g(x) = (x-2) - 6 = x - 8,
  \]
  \[
  (fg)(x) = f(x)g(x) = (x-2)(6) = 6x - 12,
  \]
  \[
  (f/g)(x) = f(x)/(g(x)) = (x-2)/6 = \frac{1}{6}x - \frac{1}{3}.
  \]

**DEFINITION.** For functions \( f \) and \( g \), define \( fo g \), the composition of \( f \) and \( g \), by

\[
(f\circ g)(x) = f(g(x))
\]

Apply \( g \) to \( x \). Get \( g(x) \). Apply \( f \) to \( g(x) \). Get \( f(g(x)) \).

\( f \) is the **outer** function; \( g \) is the **inner** function.

- Suppose \( f(x) = x-2 \) and \( g(x) = x^2 \).

  **(a)** Find \( f \circ g \) and \( g \circ f \).
  \[
  (f \circ g)(x) = f(g(x)) = f(x^2) = x^2 - 2,
  \]
  \[
  (g \circ f)(x) = g(f(x)) = g(x-2) = (x-2)^2 = x^2 - 4x + 4.
  \]

  Note that \( f \circ g \neq g \circ f \). For composition, order matters.

  **(b)** Find \( f(g)(2) \).
  Since \( f(g(x)) = x^2 - 2 \), \( f(g)(2) = 2^2 - 2 = 2 \).

  **(b')** Find \( f \circ g \) and \( g \circ f \) directly without \( f \circ g \),
  \[
  (g \circ f)(2) = g(2) = 4,
  \]
  \[
  (f \circ g)(2) = f(4) = 4 - 2 = 2.
  \]

- If \( h(x) = c \), then \( h(8) = c \), \( h(\frac{1}{2}) = c \), \( h(x^2 - 1) = c \), \( h(g(x)) = c \).

- For \( f(x) = 3x + 4 \), \( g(x) = 5 \), find \( f \circ g \) and \( g \circ f \).

  \[
  f(5) = 3 \cdot 5 + 4 = 19,
  \]
  \[
  g(5) = 3 \cdot 4 + 5 = 17.
  \]

For \( f \) and \( g \) above, note that

\[
\begin{array}{c}
\text{f(-1)} = 2, \quad \text{f(2)} = 3, \quad \text{f(4)} = 2, \\
\text{g(-2)} = -1, \quad \text{g(1)} = -2, \quad \text{g(1)} = -2, \quad \text{g(2)} = 2.
\end{array}
\]

Find

\[
\begin{array}{c}
(f \circ g)(2) = f(g(2)) = f(4) = 2, \\
(g \circ f)(2) = g(f(2)) = g(3) = 5, \\
(f \circ f)(2) = f(f(2)) = f(3) = 3.
\end{array}
\]

- \( f(x) = \frac{x + \frac{1}{2}}{x} \), \( f(f(x)) = \frac{x + \frac{1}{2}}{x} + 1/(x + \frac{1}{2}) = x + \frac{1}{2} + \frac{x}{x^2 + 1} \)

Write each function below as a composition of two simpler functions, an **outer** function \( f \) and an **inner** function \( g \).

Find the inner function first.

- Write \( x^2 + 2 \) as a composition \( f(g(x)) \).
  \[
  \begin{array}{c}
  (x^2 + 2) \quad \text{inner function } g(x) = x^2 + 2 \\
  \downarrow \\
  \text{outer function } f(x) \text{ does what remains } x^6 \text{ to be done.} \quad \therefore \quad
  f(x) = x^6, \\
  \text{check: } f(g(x)) = f(x^2 + 2) = (x^2 + 2)^6.
  \end{array}
  \]

- Write \( 4 \frac{1}{2} + 3 \) as a composition \( f(g(x)) \).
  \[
  \begin{array}{c}
  4 \frac{1}{2} + 3 \quad \text{inner function } g(x) = \frac{1}{x} \\
  \downarrow \\
  \text{outer function } f(x) \text{ does what remains } 4x + 3 \text{ to be done.} \quad \therefore \\
  f(x) = 4x + 3, \\
  \text{check: } f(g(x)) = f(\frac{1}{x}) = 4 + \frac{1}{x} + 3.
  \end{array}
  \]

- Write \( \sqrt{x + 1} \) as a composition.
  \[
  \begin{array}{c}
  \sqrt{x + 1} \quad \text{outer function } f(x) \text{ does what remains } \sqrt{x} \text{ to be done.} \quad \therefore
  \sqrt{x} \quad \text{inner function } g(x) = x + 1 \\
  \downarrow \\
  f(x) = \sqrt{x}, \\
  \text{check: } f(g(x)) = f(x + 1) = \sqrt{x + 1}.
  \end{array}
  \]

- Write \( x^4 + x^2 + 1 \) as a composition.
  \[
  \begin{array}{c}
  x^4 + x^2 + 1 = (x^2)^2 + (x^2) + 1 \\
  \text{outer function } f(x) \text{ does what remains } g(x) = x^2 \\
  f(x) = x^2 + x + 1, \\
  \text{check: } f(g(x)) = f(x^2) = x^4 + x^2 + 1.
  \end{array}
  \]

- Write \( \sqrt{\frac{x}{1 + \sqrt{x}} } \) as a composition of 2 functions.

- Write \( 1/(1 + |x|) \) as a composition of 3 functions.
  \[
  \begin{array}{c}
  \text{Ans: } h(f(g(x))), \quad g(x) = |x|, \quad f(x) = 1 + x, \quad h(x) = 1/x
  \end{array}
  \]

**DEFINITION.** \( id(x) = x \) is called the **identity** function.

Hence \( id(5) = 5 \), \( id(y) = y \), \( id(x^2 - 1) = x^2 - 1 \), ...

**THEOREM.** For any function \( f(x) \), \( f \circ id = f \) and \( id \circ f = f \).

**PROOF.** \( (f \circ id)(x) = f(id(x)) = f(x) \).

\[
(id \circ f)(x) = id(f(x)) = f(x).
\]

0 is the identity for addition, since \( f + 0 = f \).

1 is the identity for multiplication, \( f \cdot 1 = f \).

\( id(x) \) is the identity for composition, since \( f \circ id = id \circ f = f \).