Math 140   Lecture 15

Angles, arcs, and radians

Recall. A circle of radius \( r \) has circumference \( 2\pi r \).
\( \pi \approx 3.14 \). Unit circle circumference = \( 2\pi r=2\pi 1=2\pi \).
\( \theta \) and \( w \) are the Greek letters “theta” and “omega”.

Definition. Suppose the vertex of an angle is at the center of a circle of radius \( r \). Let \( s \) be the length of the arc the angle intercepts on the circle. Then
\[ \theta = \frac{s}{r} \]
is the radian measure of the angle. For unit circles, the radius \( r = 1 \) and radian measure equals arc length: \( \theta = s \).

- Radian and degree measures on the unit circle.

\[
\begin{array}{c|c|c}
\theta & \text{radians} & \text{degrees} \\hline
\frac{\pi}{2} & 90^\circ & \\
\frac{\pi}{4} & 45^\circ & \\
\pi & 180^\circ & \\
-\frac{\pi}{4} & -45^\circ & \\
-\pi & -180^\circ & \\
180^\circ & \pi & \\
0^\circ & 0 & \\
360^\circ & 2\pi & \\
\end{array}
\]
Clockwise angles are negative.

Conversion formulas. 180° = \( \pi \) radians. Thus
1° = \( \pi /180 \) radians; 1 radian = \( 180/\pi \) degrees.

- Convert 100° to radians.
\[ 100^\circ = \frac{100\pi}{180} \text{ radians} = \frac{5\pi}{9} \text{ radians} \approx 1.74 \text{ radians} \]
- Convert \( \pi/6 \) radians to degrees.
\[ \frac{\pi}{6} \text{ radians} = \frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = 30^\circ. \]

When \( \theta \) is in radians, we can solve \( \theta = s/r \) for arclength:
\[ s = \theta r \]

- Find the length of a 30° arc on a circle of radius 12 inches.
First convert to radians. By the above 30° = \( \pi/6 \) radians. \( \therefore \ s = \theta r = \frac{\pi}{6} \cdot 12 = 2\pi \) inches.

- Find the degree measure of an angle which intercepts a 5 inch arc on a circle of radius 12 inches.
\[ \theta = \frac{5}{12} \text{ radians} = \frac{5}{12} \cdot \frac{180^\circ}{\pi} = \frac{75^\circ}{\pi} \]

Speed

Definition. If an object travels a distance \( d \) in time \( t \), its linear speed is \( d/t \).

If an object rotates through an angle \( \theta \) in time \( t \), its rotational speed is \( \omega = \theta/t \).

Theorem. If a point rotates around a circle of radius \( r \) with rotational speed \( \omega \), then its linear speed is \( \omega r \).

Proof. If a point rotates through an angle \( \theta \) on a circle of radius \( r \) in time \( t \), then \( \omega = \theta/t \) and the distance \( d \) it travels = the length of the arc it traces = \( \theta r \). \( \therefore \) its linear speed = \( d/t = \theta r/t = (\theta/t)r = \omega r \). We assume \( \theta \) is in radians.

- A point revolves around a circle of radius 3 feet at 10 revolutions per minute.

(a) What is its rotational speed (in radians)?
\[ 1 \text{ revolution} = 2\pi \text{ radians}, \quad \omega = 10 \text{ revs/min} = 10 \cdot 2\pi \text{ radians/min} = 20\pi \text{ radians/min}. \]

(b) What is its linear speed? Its linear speed = \( \omega r = (20\pi \text{ rad/min}) \cdot (3 \text{ feet}) = 60\pi \text{ feet/min}. \)

Radian and degree measures on the unit circle.

Angles, arcs, and radians

When \( q \) and \( w \) are the Greek letters “theta” and “omega”.

- The six trigonometric functions of \( \theta \) are:

\[
\begin{array}{llll}
\sin \theta = y & \cos \theta = x & \csc \theta = 1/\sin \theta & \sec \theta = 1/\cos \theta \\
\tan \theta = \sin \theta / \cos \theta & \cot \theta = \cos \theta / \sin \theta & \cot \theta = \csc \theta / \sec \theta \\
\end{array}
\]

- Draw an angle of \( \pi/2 \) radians in standard position. Find the six trigonometric functions.
\[ \sin(\pi/2) = 1 \quad \csc(\pi/2) = 1 \\
\cos(\pi/2) = 0 \quad \sec(\pi/2) = \text{undef} \\
\tan(\pi/2) = \text{undef} \quad \cot(\pi/2) = 0 \]

- A point \((x, y)\) on the unit circle is in the second quadrant and \( y = \frac{3}{4} \). Find the six trig functions for \( \theta \).
\[ (x, y) \text{ on the unit circle } \Rightarrow x^2 + y^2 = 1. \]
\[ x^2 = 1 - y^2 = 1 - \left(\frac{3}{4}\right)^2 = 1 - \frac{9}{16} = \frac{7}{16}. \]
\[ x = \pm \sqrt{7}/4. \]
\[ (x, y) \text{ in the second quadrant } \Rightarrow x \text{ is negative.} \]
\[ x = -\sqrt{7}/4. \]
\[ \sin \theta = 3/4 \quad \csc \theta = 4/3 \]
\[ \cos \theta = -\sqrt{7}/4 \quad \sec \theta = -4/\sqrt{7} \]
\[ \tan \theta = -3/\sqrt{7} \quad \cot \theta = -\sqrt{7}/3 \]