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Math 140  Lecture 1

Intervals, absolute value, domains, polynomials

**Interval Notation.**

- $x \in [1, 3]$ iff $1 \leq x < 3$
- $x \in (1, 3)$ iff $1 < x < 3$
- $x \in (1, \infty)$ iff $1 < x$
- $x \in (-\infty, -1] \cup [1, \infty)$ iff $x \leq -1$ or $1 \leq x$

**Absolute Value.**

- $|3| = 3$, $|-3| = 3$, $|0| = 0$.
- Negating a negative makes it positive, $-(-3) = 3$. In general $|x| = x$ if $x \geq 0$.
- $|x| = -x$ if $x < 0$.

- **Rewrite without $| |$'s:**
  - $|\pi - \sqrt{2}| = \pi - \sqrt{2}$ since $\pi > \sqrt{2}$ implies $\pi - \sqrt{2} > 0$.
  - $|\sqrt{2} - \pi| = -(\sqrt{2} - \pi) = \pi - \sqrt{2}$.

- **Rewrite without $| |$'s:** $|x - 3| + |x - 5|$ for $x \in (3, 5)$.
- $x \in (3, 5) \Rightarrow x > 3 \Rightarrow x - 3 > 0 \Rightarrow |x - 3| = (x - 3)$.
- $x \in (3, 5) \Rightarrow x - 5 < 0 \Rightarrow |x - 5| = -(x - 5) = -x + 5$.

**Distance.** $|a - b|$ is the distance between $a$ and $b$.
- The distance between 7 and 3 is $|7 - 3| = 4$.
- The distance between 7 and 3 is $|3 - 7| = |-4| = 4$.

**Note.**

- $|x| < 3$ iff $-3 < x < 3$ if $x \in (-3, 3)$.
- $|x| > 3$ iff $x < -3$ or $3 < x$ if $x \in (-\infty, -3) \cup (3, \infty)$.

- **Rewrite as an interval:** $\{x : |x + 7| < 3\}$
- $|x + 7| < 3$ iff $-3 < x + 7 < 3$ iff $-10 < x < -4$.
- Answer: $\{x : |x + 7| < 3\} = (-10, -4)$.

- **Rewrite as a union of two intervals:** $\{x : |x - 5| \geq 2\}$
- $|x - 5| \geq 2$ iff $x - 5 \leq -2$ or $2 \leq x - 5$.
- Answer: $\{x : |x - 5| \geq 2\} = (-\infty, 3] \cup [7, \infty)$.

**Definition.** The domain of a function (real values, real variables) is the set of numbers for which it is defined.

- **Where is $\frac{x - 1}{(x - 2)(x - 3)}$ not defined?** Its domain is $x \neq 2, 3$.
- **Where is $\sqrt{x - 2}$ not defined?** Its domain is $x \geq 2$.

**Know what the following terms mean:** polynomial, degree, coefficients, quadratic function, expanded form, factored form mean. Know how to factor a polynomial and use the quadratic formula.

- $(x + 1)^2 = x^2 + 2x + 1$
- $(x + 1)^2$ is the factored form,
- $x^2 + 2x + 1$ is the expanded form.

**Solving Inequalities.**

As with equalities, you may add or subtract anything from both sides. You may multiply or divide both sides by a positive number. If you multiply or divide by a negative number, you must change the direction of the inequality (multiplying $5 \times 3$ by $-1$ gives $-5 < -3$). If you don’t know if a number is positive or negative, don’t multiply or divide by it.

- $2x \leq 4 \Rightarrow x \leq 2$ (divide by 2)
- $-2x \leq 4 \Rightarrow x \geq -2$ (divide by $-2$; change sign direction)
- $x < \frac{1}{2}$ doesn’t imply $x^2 < 1$ (don’t know if $x \geq 0$ or $x \leq 0$).

To solve problems like $x < \frac{1}{2}$, use the key-number method:

- **Rewrite with 0 on the right.** Factor the $f(x)$ on the left.
  - $f(x) < 0$, $f(x) > 0$, $f(x) \leq 0$, or $f(x) \geq 0$.
- **Find key numbers** $x$ where $f(x)$ or $f(x)$ is undefined.
- **On the key intervals** before, between, and after key numbers, $f(x)$ is either $>0$ or $<0$. To find out which, evaluate $f(x)$ at some point inside the interval. You don’t have to find the value, just the sign.

- **Use ($$’s with $<$, with $>$, around $\pm$,$\infty$, and where $f(x)$ is undefined.
- **Use [ ]’s if $f(x)$ is defined and the inequality is $\leq$ or $\geq$.

- **Solve for $x$: $x < \frac{1}{x}$.** Write as a single interval.
  - $x < \frac{1}{x}$
  - $x - \frac{1}{x} < 0$
  - $x^2 - 1 < 0$
  - $f(x) = \frac{(x - 1)(x + 1)}{x}$
  - **Key numbers:** $x = -1, 0, 1$.
  - **Key intervals:** $(\infty, -1), (-1, 0), (0, 1), (1, \infty)$.
  - $f(-2) = -$, $f(-\frac{1}{2}) = +$, $f(\frac{1}{2}) = -$, $f(2) = +$.
  - We picked $-2e(-\infty, -1), -\frac{1}{2}e(-1, 0), \frac{1}{2}e(0, 1), 2e(1, \infty)$.
  - $x < 1/x$ iff $f(x) < 0$.
  - Answer: $x \in (-\infty, -1) \cup (0, 1)$.

- **Solve for $x$: $x^2 + x - 6 \leq 0$.** Put answer in interval notation.
  - 1st factor.
  - $f(x) = x^2 + x - 6 = (x - 2)(x + 3)$.
  - $f(x) = 0$ iff $x = 2, -3$. These are the key numbers.
  - The 3 key intervals: $(-\infty, -3), [-3, 2], (2, \infty)$.
  - Picking $-4, 0, 3$: $f(-4)>0$, $f(0)<0$, $f(3)>0$.
  - $x^2 + x - 6 \leq 0$ iff
  - Answer: $x \in [-3, 2]$.
Rectangular coordinates

\[(a,b)\] is the point on the plane whose x-coordinate is \(a\) and y-coordinate is \(b\).

**WARNING.** \((a,b)\) can mean a point or an interval.

**Definition.** The graph of an equation is the set of all points which satisfy the equation.

The graph of \(y=0\) is the x-axis.
The graph of \(x=0\) is the y-axis.

- Does \((5,6)\) lie on \(y=x^2+1\)?
  \((x,y)\)=(5,6) lies on \(y=x^2+1\) iff \(6=(5)^2+1\) iff \(6=26\) iff no.

**Distance formula.** The distance \(d\) between points \((x_1, y_1)\) and \((x_2, y_2)\) is
\[
d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}
\]
- The distance between \((1,3)\) and \((2,5)\) is
  \[
  \sqrt{(1-2)^2 + (3-5)^2} = \sqrt{1 + 4} = \sqrt{5}
  \]

**Circles and completing the square**

**Circle equation.** The circle with radius \(r\) and center \((a,b)\) has equation:
\[
(x-a)^2 + (y-b)^2 = r^2.
\]
Proof. \((x,y)\) is on this circle iff the distance between \((x,y)\) and \((a,b)\) is \(r\).
- iff \(\sqrt{(x-a)^2 + (y-b)^2} = r\)
- iff \((x-a)^2 + (y-b)^2 = r^2\).

- Find the equation for the circle of radius 3 around \((1,2)\):
  \((x-1)^2 + (y-2)^2 = 3^2\)

**Facts.** For positive \(a\), \(a = (\sqrt{a})^2\). For all \(a\), \(\sqrt{a^2} = |a|\).
- Find the center and radius of the circle with equation:
  \((x+1)^2 + y^2 = 8\)
  \((x-(-1))^2 + (y-0)^2 = (\sqrt{8})^2 = (\sqrt{2 \cdot 4})^2 = (2\sqrt{2})^2\)
  \(\therefore\) center = \((-1,0)\), radius = \(2\sqrt{2}\)

**Completing the square.** To make \(x^2 + ax\) a perfect square, add \((\frac{a}{2})^2\).
\[x^2 + ax + \left(\frac{a}{2}\right)^2 = (x + \frac{a}{2})^2\]. The latter is a “perfect” square.
- Find the center and radius of the circle with equation:
  \(4x^2 + 4y^2 + 4y - 79 = 0\)
  \[x^2 + y^2 + y - \frac{79}{4} = 0\]
  \[x^2 + y^2 + y = \frac{79}{4}\]
  \[(x-0)^2 + (y+\frac{1}{2})^2 = \frac{79}{4} + \frac{1}{4} = \frac{80}{4} = 20\]
  \[(x-0)^2 + (y+\frac{1}{2})^2 = (\sqrt{20})^2 = (\sqrt{4 \cdot 5})^2\]
  \[(x-0)^2 + (y-(-\frac{1}{2}))^2 = (2\sqrt{5})^2\]
  \(\therefore\) center = \((0,-\frac{1}{2})\), radius = \(2\sqrt{5}\)

**Definition.** An x-intercept is the x-coordinate of a point where the graph crosses the x-axis. A y-intercept is the y-coordinate of a point where the graph crosses the y-axis. In fig. 2, \(2\) is the y-intercept; \(1,5\) are the x-intercepts.

The graph crosses the x-axis when the y-coordinate is 0.
To find the x-intercepts, set \(y\) to 0 and solve for \(x\).
To find the y-intercepts, set \(x\) to 0 and solve for \(y\).

- Find the x and y-intercepts.
  \(y - x^2 + 4 = 0\)
  x-intercepts (set \(y\)=0):
  \(-x^2 + 4 = 0\), \(-x^2 = -4\), \(x^2 = 4\), \(x = -2, 2\)
  The x-intercepts are \(-2, 2\).
  y-intercepts (set \(x\)=0):
  \(-y^2 + 4 = 0\), \(y = -4\)
  The y-intercept is \(-4\).

- Find the x and y-intercepts.
  \(y - x^2 - 4 = 0\)
  x-intercepts:
  \(-x^2 - 4 = 0\), \(-x^2 = 4\), \(x^2 = -4\), x-intercepts: none
  y-intercept: \(4\).

**Straight lines: slopes and equations**

**Definition.** The slope of a line is the ratio of the vertical (height) change in \(y\) over a horizontal change in \(x\).

**Theorem.**
- The slope of the line through \((x_1,y_1)\) and \((x_2,y_2)\):
  \(m = \frac{y_2-y_1}{x_2-x_1}\)
- The equation of the line through \((x_1,y_1)\) with slope \(m\): \(y - y_1 = m(x-x_1)\)
- The equation of the line with slope \(m\) and y-intercept \(b\): \(y = mx + b\)
- The slope of a horizontal line is 0.
- The slope of a vertical line is undefined.
- Equation for the horizontal line through \((a,b)\): \(y = b\).
- Equation of the vertical line through \((a,b)\): \(x = a\).

**Line equation format.** In homework and tests, line equations must be in one of these four forms:
\(y = mx + b\), \(y = mx\), \(y = b\), \(x = a\).
- Find the line through \((2, 5)\) and \((4, 1)\). Check your answer.
  \(m = (5-1)/(2-4) = -2\), \(y - 5 = -2(x-2)\), \(y = -2x + 9\)
- Eq. of line with slope -8, y-intercept -6. \(y = -8x - 6\)
Math 140  Lecture 3

Reminder: Gateway Exam week from Thursday.

**Factoring and roots**

**Theorem.** If \( a > 0 \), \( x^2 - a = (x - \sqrt{a})(x + \sqrt{a}) \). But \( x^2 + a \) has no roots and can’t be factored any more.

**Division Law.** If \( p(x)/d(x) \) has quotient \( q(x) \) and remainder \( r(x) \) then \( p(x)/d(x) = q(x) + r(x)/d(x) \). Multiply by \( d(x) \) to get \( p(x) = d(x)q(x) + r(x) \). If \( d(x) \) divides \( p(x) \) evenly iff the remainder is 0 iff \( p(x) = d(x)q(x) \) iff \( d(x) \) is a factor of \( p(x) \).

If \( d(x) \) is a factor of \( p(x) \), the other factor of \( p(x) \) is the quotient factor \( q(x) \). To get this quotient factor divide: \( p(x)/d(x) \).

- Given \( p(x)/d(x) \), divide to get the quotient \( q(x) \) and remainder \( r(x) \). Write the answer in division law form: \( p(x) = d(x)q(x) + r(x) \).

\[
\frac{x^4 - 1}{x - 1} = (x^2 + x + 1 + \frac{2}{x-1}) = (x-1)(x^2 + x + 1) + 2
\]

Check that the answer is correct for \( x=0 \). For \( x=0 \), we get 0+1=(-1)(0+0+1)+2, 1=1. ✓

**X-Intercept.** \( a \) is a root or zero of \( p(x) \) iff \( p(a) = 0 \).

**Theorem.** \( a \) is a root of \( p(x) \) iff \( (x-a) \) is a factor of \( p(x) \).

To find all roots of \( p(x) \), completely factor \( p(x) \).

**Factor the polynomial and find all roots.**

- \( x + 2 \) Root: -2
- \( x^2 + 2 \) Fully factored as is, no roots.
- \( x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2}) \) Roots: \( -\sqrt{2,} \sqrt{2} \)
- \( x^2 - 4x + 4 = (x - 2)^2 \) One repeated factor. Root: 2
- \( x^3 + 5x^2 + 8x + 4 \) given that \(-1\) is a root.
  \((x - (-1)) = (x + 1) \cdot \cdot \cdot \) we divide by \(x+1\).
  \[
  \frac{x^3 + 5x^2 + 8x + 4}{x + 1} = x^2 + 4x + 4 = (x+2)(x+2)
  \]
  \[
  x^3 + 5x^2 + 8x + 4 = (x+1)(x+2)^2
  \]
  Roots: -1, -2.
- \( x^3 - x^2 - 2x + 2 \) given that 1 is a root.
  \[
  \frac{x^3 - x^2 - 2x + 2}{x - 1} = x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})
  \]
  \[
  x^3 - x^2 - 2x + 2 = (x-1)(x + \sqrt{2})(x - \sqrt{2})
  \]
  Roots: 1, \( -\sqrt{2,} \sqrt{2} \).
- \( 2x^2 + 2x - 2 \). Factor out the coefficient of \( x^2 \); find the roots with the quadratic formula; factor.
  \[
  2(x^2 + x - 1)
  \]
  Roots: \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2} \)
  Factorization: \( 2(x - \frac{-1 + \sqrt{5}}{2})(x - \frac{1 - \sqrt{5}}{2}) \)

**Functions**

**Definition.** For sets \( A \) and \( B \), a function from \( A \) to \( B \) assigns a value \( f(x) \) in \( B \) to each \( x \) in \( A \). The domain of \( f \) is \( A \); the range of \( f \) is the set of all possible values \( f(x) \).

\( f(x) = x^2 \) is a function from real numbers to real numbers.

- Domain = \( (-\infty, \infty) \) since \( x^2 \) is defined for all numbers.
- Range = \( [0, \infty) \) since \( x^2 \) can never be negative.

**Notation.** Sometimes, instead of writing \( f(x) = x^2 \), we define a function by writing \( y = x^2 \).

Thus \( y \) is the value of the function. Since it depends on \( x \), \( y \) is the dependent variable. Since \( x \) ranges freely over the domain, it is the independent variable.

A function may assign only one value to each \( x \). Thus \( y = \pm \sqrt{x} \) is not a function.

- Of \( f \) and \( g \), which are functions? (\( f \) isn’t, \( g \) is)
  - \( y = 1 - x \) \( (-\infty, \infty) \)
  - \( y = \frac{1}{1 - x} \) \( (-\infty, 1) \cup (1, \infty) \)
  - \( y = \sqrt{1 - x} \) \( (-\infty, 1] \)

- \( f(x) = x^2 \). Simplify to an expanded polynomial.
  - \( f(2) = 4 \)
  - \( f(x) + 2 = x^2 + 2 \)
  - \( xf(x) = x^3 \)
  - \( f(x^2) = x^6 \)
  - \( (f(x))^3 = x^6 \)
  - \( f(f(x)) = x^4 \)
  - \( f(x) - f(a) \)
    \[
    \frac{x - a}{x - a} = \frac{x^2 - a^2}{x - a} = x + a
    \]
  - \( f(x + h) - f(x) \)
    \[
    \frac{(x + h)^2 - x^2}{h} = \frac{2x + h}{h} = 2x + h
    \]
**Math 140  Lecture 4**

**Gateway exam.** Gateway practice exams included with lecture notes.

**Definition.** The graph of a function $f$ is the set of all points $(x, y)$ such that $y = f(x)$. The height $y$ is the function’s value $f(x)$.

**Fact.** A curve is the graph of a function iff no vertical line intersects it more than once.

- Which curve is the graph of a function?

- Find the domain and range of $f$, $g$, and $h$.
  - Domain: $[-1, 3]$  
  - Range: $[1, 4]$  
  - Find the values. $f(1) = 4$  
  - For what value of $x$ is $f(x) = 1$? $x = 2$  
  - Compute $[f(x) - f(1)]/[x-1]$ when $x = 3$. $\frac{-1}{2}$

  - Domain: $(-1, 3]$  
  - Range: $[2]$  
  - Find the values. $g(1) = 2$  
  - For what value of $x$ is $g(x) = 2$? $x = 2$

  - Domain: $[0, 2) \cup (2, 4]$  
  - Range: $[1, 3) \cup (4, 5]$  
  - Find the values. $h(2) = \text{undefined}$  
  - $h(4) = 1$

**Definition.**
- $f$ increases on an interval if the value $f(x)$ increases as $x$ moves from left to right in the interval.
- $f$ decreases if $f(x)$ decreases as $x$ goes left to right.
- $c$ is a turning point of $f$ if $f$ increases on one side of $c$ and decreases on the other.
- $f(a) = c$ is a maximum value iff $c$ is $\geq$ all other values.
- $f(a) = c$ is a minimum value iff $c$ is $\leq$ all other values.

- For $f$ and $h$ graphed above, fill in the table. Write “none” if there are none.

<table>
<thead>
<tr>
<th></th>
<th>$f$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>turning points</td>
<td>$x = 1, 2$</td>
<td>none</td>
</tr>
<tr>
<td>maximum value</td>
<td>$f(1) = 4$</td>
<td>$h(0) = 5$</td>
</tr>
<tr>
<td>minimum value</td>
<td>$f(2) = 1$</td>
<td>$h(4) = 1$</td>
</tr>
<tr>
<td>interval(s) of increase</td>
<td>$[-1, 1], [2, 3]$</td>
<td>none</td>
</tr>
<tr>
<td>interval(s) of decrease</td>
<td>$[1, 2]$</td>
<td>$[0, 2), (2, 4]$</td>
</tr>
</tbody>
</table>

- Do the same for the functions $f$ and $g$ below.

<table>
<thead>
<tr>
<th></th>
<th>$f$</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>domain</td>
<td>$[-2, 5]$</td>
<td>$[0, 4]$</td>
</tr>
<tr>
<td>range</td>
<td>$[-1, 4]$</td>
<td>$[0, 3]$</td>
</tr>
<tr>
<td>turning points</td>
<td>$x = 0, 2$</td>
<td>$x = 2$</td>
</tr>
<tr>
<td>maximum value</td>
<td>$f(0) = 4$</td>
<td>$g(2) = 3$</td>
</tr>
<tr>
<td>minimum value</td>
<td>$f(2) = -1$</td>
<td>$g(0) = g(4) = 0$</td>
</tr>
<tr>
<td>interval(s) of increase</td>
<td>$[-2, 0], [2, 5]$</td>
<td>$[0, 2]$</td>
</tr>
<tr>
<td>interval(s) of decrease</td>
<td>$[0, 2]$</td>
<td>$[2, 4]$</td>
</tr>
</tbody>
</table>

- Fill in the table.

<table>
<thead>
<tr>
<th></th>
<th>$f(x) = \sqrt{x^2 - 1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>domain</td>
<td>$(\infty, -1] \cup [1, \infty)$</td>
</tr>
<tr>
<td>range</td>
<td>$[-1, \infty)$</td>
</tr>
<tr>
<td>turning points</td>
<td>none</td>
</tr>
<tr>
<td>maximum value</td>
<td>none</td>
</tr>
<tr>
<td>minimum value</td>
<td>none</td>
</tr>
<tr>
<td>interval(s) of increase</td>
<td>$f(-1) = f(1) = 0$</td>
</tr>
<tr>
<td>interval(s) of decrease</td>
<td>$[1, \infty)$</td>
</tr>
</tbody>
</table>

- Graph.

- Graph.

- Gateway problem. Complete the square of the formula.

$$2x^2 - 3x + 5 = \ldots = 2(x - \frac{3}{4})^2 + \frac{31}{8}$$
Basic graphs
Know these graphs.

\[ 1, \quad x, \quad x^2, \quad x^3, \quad 1/x. \]

**Translations and reflections**

**Theorem.** Translating a graph \( f(x) \) or reflecting it across an axis, changes the function as follows:

<table>
<thead>
<tr>
<th>up 1 unit</th>
<th>down 1</th>
<th>*left 1</th>
<th>*right 1</th>
<th>reflect in x-axis</th>
<th>reflect in y-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x)+1 )</td>
<td>( f(x)-1 )</td>
<td>( f(x+1) )</td>
<td>( f(x-1) )</td>
<td>-( f(x) )</td>
<td>( f(-x) )</td>
</tr>
</tbody>
</table>

* Horizontal changes are the opposite of what one would expect.

- Given \( f(x) \), find the functions for the other graphs.

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f(x) )</th>
<th>( f(x+1) )</th>
<th>( f(x-1) )</th>
<th>-( f(x) )</th>
<th>( f(-x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( x )</td>
<td>( x+1 )</td>
<td>( x-1 )</td>
<td>-( x )</td>
<td>( -x )</td>
</tr>
</tbody>
</table>

- **Horizontal moves**

\( x = \) the x-axis position. Changing \( x \), changes the horizontal position of the coordinate system. Replacing \( x \) by \( x+2 \) shifts the coordinate system 2 units to the right.

Moving the coordinate system to the right equals moving the graph to the left. Stated in terms of the graph rather than the coordinate system, changing \( x \) changes the position of the graph in the opposite direction.

Replacing \( x \) by \( x+2 \) shifts the graph left 2 units.
Replacing \( x \) by \( x-2 \) shifts the graph right 2 units.
Replacing \( x \) by \( -x \) reflects the graph horizontally across the y-axis.

- **Vertical moves**

\( f(x) = \sqrt{x} . \) Describe the following shifts.

\[ \begin{align*}
\text{As above} & \\
\text{Shift right 1} & \\
\text{Shift left 1} & \\
\text{Reflect in y-axis} & \\
\text{Shift left 1, reflect in y-axis} & \\
\text{Shift up 1} & \\
\text{Reflect in x-axis} & \\
\text{Shift left 1, reflect across y-axis, reflect across x-axis, up 1.} & \\
\end{align*} \]

- **Given \( f(x) = |x| \), graph \( f(x)+2, f(x)-2, -f(x) \).**

\[ f(x)=|x|, \quad f(x)+2=|x|+2, \quad f(x)-2=|x|-2, \quad -f(x)=-|x|. \]

- **Given \( f(x) = x^2 - x \).** Graph

\[ g(x) = x^2 - x. \]

\[ \begin{align*}
\text{Graph} & \\
\text{Parabola with roots 0, 1} & \\
\text{Shift left 1} & \\
\text{Shift left 1, reflect in y-axis} & \\
\end{align*} \]
Math 140 Lecture 6
Study Practice Exam 1 and the recommended exercises.

Functions can be added and multiplied just like numbers.

**DEFINITION.** For functions $f$ and $g$, define $f+g, f-g, fg, f/g$
by
\[
(f+g)(x) = f(x) + g(x),
\]
\[
(f-g)(x) = f(x) - g(x),
\]
\[
(fg)(x) = f(x)g(x),
\]
\[
(f/g)(x) = f(x)/g(x).
\]

- If $f(x) = x-2$, $g(x) = 6$, then
  \[
  (f+g)(x) = f(x)+g(x) = (x-2)+(6) = x+4,
  \]
  \[
  (f-g)(x) = f(x)-g(x) = (x-2)-(6) = x-8,
  \]
  \[
  (fg)(x) = f(x)g(x) = (x-2)(6) = 6x-12,
  \]
  \[
  (f/g)(x) = f(x)/g(x) = (x-2)/6 = \frac{1}{6}x - \frac{1}{3}.
  \]

**DEFINITION.** For functions $f$ and $g$, define $f \circ g$, the
composition of $f$ and $g$, by
\[
(f \circ g)(x) = f(g(x))
\]
Apply $g$ to $x$. Get $g(x)$. Apply $f$ to $g(x)$. Get $f(g(x))$.
$f$ is the outer function; $g$ is the inner function.

- Suppose $f(x) = x-2$ and $g(x) = x^2$.
  (a) Find $f \circ g$ and $g \circ f$.
  \[
  (f \circ g)(x) = f(g(x)) = f(x^2) = x^2-2.
  \]
  \[
  (g \circ f)(x) = g(f(x)) = g(x-2) = (x-2)^2 = x^2-4x+4.
  \]
  Note that $f \circ g \neq g \circ f$. For composition, order matters.
  (b) Find $(f \circ g)(2)$.
  \[
  (f \circ g)(2) = f(g(2)) = f(2^2) = f(4) = 4-2 = 2.
  \]
  \[
  (g \circ f)(2) = g(f(2)) = g(2-2) = g(0) = 0^2 = 0.
  \]

- If $h(x) = c$, then $h(8) = c$, $h(y) = c$, $h(x^2-1) = c$, $h(g(x)) = c$.
- For $f(x) = 3x+4$, $g(x) = 5$, find $f \circ g$ and $g \circ f$.
  \[
  f(g(x)) = f(5) = 3 \cdot 5 + 4 = 19.
  \]
  \[
  g(f(x)) = g(3x+4) = 5.
  \]

For $f$ and $g$ above, note that
\[
 f(-3) = 1, f(-1) = 2, f(2) = 3, f(4) = 2,\]
and $g(-2) = -1, g(-1) = -2, g(1) = -1, g(2) = 2$.

Find
\[
 (f \circ g)(2) = f(g(2)) = f(2) = 3.
\]
\[
 (g \circ f)(2) = g(f(2)) = g(4) = 4-2 = 2.
\]
\[
 (f \circ f)(-1) = f(f(-1)) = f(2) = 3.
\]
\[
 f(x) = x + \frac{1}{x}, f(f(x)) = (x + \frac{1}{x}) + 1/(x + \frac{1}{x}) = x + \frac{1}{x} + \frac{x}{x^2+1}.
\]

Write each function below as a composition
$f(g(x))$ of two simpler functions, an outer function $f$ and an inner function $g$.

Find the inner function first.
- Write $(x^2+2)^6$ as a composition $f(g(x))$.
  $(x^2+2)^6$ inner function $g(x) = x^2 + 2$
  outer function $f(x)$ does what remains to be done. \[ f(x) = x^6 \]
  check: $f(g(x)) = f(x^2 + 2) = (x^2 + 2)^6$.
- Write $4/3 + 3$ as a composition $f(g(x))$.
  $4/3 + 3$ inner function $g(x) = 1/x$
  outer function $f(x)$ does what remains to be done. \[ f(x) = 4x + 3 \]
  check: $f(g(x)) = f(1/x) = 4(1/x) + 3$.
- Write $\sqrt{x+1}$ as a composition.
  $\sqrt{x+1}$
  \[
  g(x) = x + 1
  \]
  \[
  f(x) = \sqrt{x}
  \]
  \[ f(g(x)) = f(x+1) = \sqrt{x+1} .
  \]
- Write $x^4 + x^2 + 1$ as a composition.
  $x^4 + x^2 + 1 = (x^2)^2 + (x^2) + 1$
  \[
  g(x) = x^2
  \]
  \[
  f(x) = x^2 + x + 1
  \]
  \[ f(g(x)) = f(x^2) = x^4 + x^2 + 1 .
  \]
- Write $\sqrt{x}/(1 + \sqrt{x})$ as a composition of 2 functions.
  Write $1/(1 + \sqrt{x})$ as a composition of 3 functions.
  \[ = h(f(g(x))), g(x) = \sqrt{x}, f(x) = 1+x, h(x) = 1/x
  \]

**DEFINITION.** $id(x) = x$ is called the identity function.

Hence $id(5) = 5, id(y) = y, id(x^2-1) = x^2-1, \ldots$

**THEOREM.** For any function $f(x)$, $f \circ id = id$ and $id \circ f = f$.

**PROOF.** $(f \circ id)(x) = f(id(x)) = f(x)$.
\[
 (id \circ f)(x) = id(f(x)) = f(x).
\]
0 is the identity for addition, since $f + 0 = 0 + f = f$.
1 is the identity for multiplication, $f \cdot 1 = 1 \cdot f = f$.
$id(x)$ is the identity for composition, since $f \circ id = id \circ f = f$. 
Inverse functions

**Definition.** $f^{-1}$, the inverse of $f$, is the function, if any, such that,
\[
f^{-1}(f(x)) = x \quad \text{when } f^{-1}(x) \text{ is defined and}
\]
\[
f^{-1}(f(x)) = x \quad \text{when } f(x) \text{ is defined.}
\]
This says that $f$ and $f^{-1}$ undo each other:
\[
f^{-1} \text{ undoes what } f(x) \text{ does and gives you back } x.
\]

- $f(x) = 2x$, $f^{-1}(x) = \frac{1}{2}x$. Verify: $f(f^{-1}(x)) = x$ & $f^{-1}(f(x)) = x$.
- $g(x) = x + 3$, $g^{-1}(x) = x - 3$.
- $h(x) = 2x + 3$, $h^{-1}(x) = \frac{x - 3}{2} = \frac{1}{2}x - \frac{3}{2}$.

Verify that $h^{-1}(h(x)) = x$.

To undo a sequence of operations, you must undo them in the reverse order: the inverse of $g(f(x))$ is $f^{-1}(g^{-1}(x))$.

Let $y = f^{-1}(x)$.
\[
f^{-1}(x) = x, \quad \text{by definition of inverse.}
\]
\[
\therefore f(y) = x, \quad \text{since } y = f^{-1}(x).
\]
The converse is also true, thus

**Theorem.** $y = f^{-1}(x)$ iff $f(y) = x$.

To find $f^{-1}(x)$ for complicated functions:

Start with $f(y) = x$.

Solve for $y$ to get $y = f^{-1}(x)$.

- $f(x) = x^3$, find $f^{-1}(x)$.
  \[
  f(y) = x
  \]
  \[
  \therefore y^3 = x
  \]
  \[
  \therefore y = \sqrt[3]{x}, \quad f^{-1}(x) = \sqrt[3]{x}
  \]

**Warning.** $f^{-1}(x)$ and $(f^{-1}(x))^{-1}$ are not the same.

- $f^{-1}(x) = \sqrt[3]{x}$
- $(f^{-1})^{-1} = \sqrt[3]{1/x^3}$
- $(f^{-1})(0) = 0$
- $(f^{-1})^{-1} = 1/x^3$
- $(f^{-1})^{-1}$ is undefined.

- $f(x) = \frac{x+1}{x-1}$, find $f^{-1}(x)$.
  \[
  f(y) = x, \quad \frac{y+1}{y-1} = x, \quad y + 1 = x(y - 1)
  \]
  \[
  y + 1 = xy - x, \quad y - xy = -x - 1
  \]
  \[
  y(1-x) = -x - 1, \quad y = \frac{-x-1}{1-x} = \frac{x+1}{x-1}
  \]
  \[
  \therefore f^{-1}(x) = \frac{x+1}{x-1}
  \]

- If $f(x) = x+3$ then $f^{-1}(x) = x-3$.
- If $g(x) = x/2$ then $g^{-1}(x) = 2x$.
- If $h(x) = \sqrt[3]{x}$ then $h^{-1}(x) = x^3$ for $x \geq 0$.

Note how the graph of $f$ is related to the graph of $f^{-1}$.

By the Theorem, $y = f^{-1}(x)$ if $f(y) = x$. Thus the graph of $y = f^{-1}(x)$ is the graph of $y = f(x)$ which is just the graph of $f(x) = y$ with $x$ and $y$ interchanged. Interchanging $x$ and $y$ reflects the plane around the major diagonal $y = x$. Hence

**Theorem.** The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the major diagonal $y = x$.

For each function, draw the three graphs $y = f(x)$, $y = x$, $y = f^{-1}(x)$ on the same coordinate system.

- $f(x) = x^3$
- $f(x) = -x^3$

**Definition.** $f$ is 1-1 (“one-to-one”) iff $x \neq y$ implies $f(x) \neq f(y)$.

- $f(x) = 3x$ is 1-1
- $f(x) = x^2$ is not
- $1\neq -1$ but $(-1)^2 = 1^2$.

**Theorem.** The following are equivalent:

- $f$ has an inverse
- $f$ is 1-1
- no horizontal line intersects its graph more than once.

- Which of the following functions has an inverse?

By the Theorem, the domain of $f^{-1}$ is the range of $f$. The range of $f^{-1}$ is the domain of $f$.

**Proof.** The reflection around the major diagonal which carries the graph of $f$ to the graph of $f^{-1}$ also carries the domain of $f$ to the range of $f^{-1}$ and the range of $f$ to the domain of $f^{-1}$.

Stated in full, the inverse is the compositional inverse. Compare it with the additive inverse and the multiplicative inverse.

For addition, 0 is the identity and the additive inverse of $f$ is the negative $-f$ since $f + (-f) = (-f) + f = 0$.

For multiplication, 1 is the identity and the multiplicative inverse of $f$ is the reciprocal $1/f$ since:
\[
f \cdot (1/f) = (1/f) \cdot f = 1.
\]

For composition, $id$ is the identity where $id(x) = x$ and for the inverse $f^{-1}$ of $f$, the corresponding equation is:
\[
f \circ f^{-1} = f^{-1} \circ f = id.
\]
Math 140 Lecture 8

**Definition.** A quadratic function is a degree-2 polynomial 
\[ y = ax^2 + bx + c \] with \( a \neq 0 \).

The graph is a parabola.
- If \( a > 0 \), the horns point up.
- If \( a < 0 \), the horns point down.
- If \( |a| > 1 \), the parabola is narrower than \( y = x^2 \).
- If \( |a| < 1 \), the parabola is wider than \( y = x^2 \).

Find the roots by factoring or using the quadratic formula:
\[
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
No roots if \( b^2 - 4ac < 0 \).

**Completing the Square Theorem.** Every quadratic function may be written in the form: 
\[ y = a(x-x_0)^2 + y_0 \]
where \((x_0, y_0)\) is the vertex (nose) of the parabola.

**Proof.** Factor the \( a \) out of the \( ax^2 + bx \) part of \( ax^2 + bx + c \).

Complete the square. Anything which is added must also be subtracted to preserve equality.

**Examples**
- Find the roots (they are the x-intercepts).
- Write in the form \( y = a(x-x_0)^2 + y_0 \).
- Graph. On the graph list both coordinates of the vertex.

\[ y = -\frac{1}{2}(x+1)^2 \]
Roots: \( x = -1 \)
\[ y = -\frac{1}{2}(x-(-1))^2 + 0 \quad \text{vertex: } (-1, 0) \]

\[ y = 2x^2 - 2x \]
\[ y = 2x^2 - 2x = 2(x-1) \]
Roots: 0, 1.
\[ y = 2(x^2 - x) \]
\[ y = 2(x^2 - x + \frac{1}{4}) - 2 \cdot \frac{1}{4} \]
The added \( 1/4 \) is multiplied by 2, thus so is the subtracted \( 1/4 \).
\[ y = 2(x - \frac{1}{2})^2 + (-\frac{1}{2}) \quad \text{vertex: } (1/2, -1/2) \]

\[ y = x^2 + 2x - 1 \]
Roots: \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2} \)
\[ y = (x^2 + 2x + 1) - 1 - 1 \]
\[ y = (x + 1)^2 - 2 \]
\[ y = (x - (1))^2 + (-2) \quad \text{vertex: } (-1, -2) \]

**Word problems**
- **Draw the picture.** Indicate the variables in the picture.
- **Write the given equations which relate the variables.**
- **Solve for the wanted quantities.** List Given and Answer.

- The perimeter of a rectangle is 10. Express the area \( A \) in terms of the width \( x \).

**Picture:**

- Given: \( 10 = 2x + 2y \)
- Answer: \( A = xy \)

- The corner of a triangle lies on the line \( y = 4-x \). Express the area \( A \) and perimeter \( P \) of the triangle in terms of the base \( x \).

**Picture:**

- Given: \( y = 4-x \)
- Answer: \( A = \frac{x(4-x)}{2} \)

- The area of an isosceles triangle is 16. Express the height of the triangle in terms of its width \( x \).

**Picture:**

- Given: \( \frac{1}{2}xh = 16 \)
- Answer: \( h = \frac{32}{x} \)
Express the height of an equilateral triangle as a function of a side $x$.

\[ h^2 + \left(\frac{x}{2}\right)^2 = x^2 \]

Want $h$ in $x$. 5 symbols

\[ h = \frac{\sqrt{3}}{2} x \]

The height of a can (right circular cylinder) is three times the radius.

Given: $h = 3r$, $S = 2\pi rh$

(a) Express the curved surface area as a function of the radius. 4 symbols

\[ S = 6\pi r^2 \]

(b) Express the radius as a function of the curved surface area. 5 symbols

\[ r = \frac{S}{6\pi} \]

Three sides of a 500 square foot rectangle are fenced. Express the length of the fence as a function of side $x$.

Given: $xy = 500$

Want $f$ in $x$, need $y$ in $x$.

\[ y = \frac{500}{x}, \quad f = 2x + \frac{500}{x} \]

**Graphing polynomials**

A polynomial graph is *smooth*: no breaks, no sharp corners.

For an expanded polynomial $ax^n + \ldots + c$ with $ax^n$ the term of highest degree: $ax^n$ is the leading term, $a$ is the leading coefficient, and $n$ is the degree. The y-intercept is the constant term $c$.

\[ y = -3x^4 + x^2 - 5, \quad \text{leading term} = -3x^4, \quad \text{leading coeff.} = -3, \quad \text{degree} = 4, \quad \text{constant term} = -5. \]

\[ y = x - x^3, \quad \text{leading term} = -x^3, \quad \text{leading coeff.} = -1, \quad \text{degree} = 3, \quad \text{constant term} = 0. \]

For large $x$ (near $\pm \infty$), graph looks like the leading term $ax^n$.

As $x$ goes to $\infty$, $y$ goes to $+\infty$ if $a > 0$, to $-\infty$ if $a < 0$.

Graphs of odd degree go to $+\infty$ in one direction, $-\infty$ in the other, like $y = x^3, y = -x^3$.

Graphs of even degree either go to $+\infty$ in both directions or to $-\infty$ in both directions, like $y = x^2, y = -x^2$.

Use the factored form to get the roots and their degrees (the *degree* of a root is the exponent of its factor).

- At roots of degree 1, the graph crosses x-axis like $y = x$ or $y = -x$.
- At roots of odd degree $n > 1$, the graph crosses the x-axis like $y = x^3$ or $y = -x^3$.
- At roots of even degree, the graph touches but doesn’t cross the x-axis, like $y = x^2$ or $y = -x^2$.

To get the leading coefficient and degree of a factored polynomial, replace each factor by its leading term.

To get the constant term (y-intercept), set $x = 0$.

When graphing, find the x and y-intercepts and also calculate a key value in each of the key intervals before, after, and between roots.

Graph $y = (x+2)(4-x)(2x-1)^2$.

-2 is a root of degree 3, 4 is a root of degree 1, ½ is a root of degree 2.

lead term = $(x)^1(-x)(2x)^2 = -4x^5$, degree = 6, lead coeff. = -4.

Constant: setting $x = 0$, \( (2)^1(4)(-1)^2 = 32 \), y-intercept = 32.

Values in the key intervals (-\$\infty$, -2], [-2, \( \frac{1}{2} \]), \( \frac{1}{2} \), 4, \( \infty \)):

\[ f(-3) = -343, \quad f(-1) = 45, \quad f(2) = 1152, \quad f(5) = -27783. \]

The dotted lines indicate the portion of the graph around $f(2) = 1152$ which exceeds our coordinate limits.
Rational functions and their graphs

**Definition.** A rational function is a ratio of two polynomials. It is reduced if the top and bottom have no common factors.

Like polynomials, rational functions have smooth graphs.

- In the graphs below, $x=0$ (the y-axis) is the vertical asymptote, $y=0$ (the x-axis) is the horizontal asymptote.

  ![Graphs of rational functions](image)

- For odd degree vertical asymptotes, one side goes to $+\infty$, the other to $-\infty$. See $1/x$ and $-1/x$.
- For even degree vertical asymptotes, both sides go to $+\infty$ or both go to $-\infty$. See $1/x^2$ and $-1/x^2$.

**Definition.** For rational functions, the leading term is the simplified ratio of the leading terms of the top and bottom.

Recall, to get the leading term of a factored polynomial, replace each factor to its leading term and then simplify.

Rational functions:

\[
\frac{x^3 + 1}{3x^2} \quad \frac{(x-1)(5x+2)}{(x-3)^2(2x+6)} \quad \frac{(1-2x)^3}{2x-6}
\]

Leading terms:

- $-1$  
- $\frac{5}{8x^3}$  
- $-4x^2$

Hor. asymptote:

- $y = -1$  
- $y = \frac{1}{3}$  
- $y = 0$  
- none

For a reduced rational function:

- x-intercepts (roots) occur where the top is 0. If the root has degree $n$, the x-intercept looks like that of $y = x^n$ or $y = -x^n$.
- If the bottom is 0 at $a$, then $x=a$ is a vertical asymptote. If the factor has degree $n$, the vertical asymptote looks like that of $y = 1/x^2$ or $y = -1/x^2$.
- As $x \to \pm \infty$, the graph resembles the graph of the leading term which is either a constant $b$ or of the form $ax^n$ or $ax^n$.
  
  (1) If a constant $b$, then $y = b$ is a horizontal asymptote.
  
  (2) If it is $ax^n$, then $y = 0$ is a horizontal asymptote.
  
  (3) If it is $ax^n$, there is no horizontal asymptote.

**Graph.** On the graph mark the x and y-intercepts. Mark the vertical and horizontal asymptotes with their equations ($y = a$ or $x = a$).

$a$ is a key number if $f(a)$ is 0 or undefined. The key intervals lie before, between and after the key numbers. Calculate a “key value” from each key interval.

**Example 1:**

\[y = \frac{2x-6}{2x^3 - 8x^2}\]

Reduce and factor:

\[
\frac{2(x-3)}{2(x^3-4x^2)} = \frac{(x-3)}{x^3-4x^2} = \frac{x-3}{x^2(x-4)}
\]

y-intercept: none

x-intercept: 3 (deg 1)

Vertical asymptotes:

\[
x = 0 \text{ (deg 2)}, \quad x = 4 \text{ (deg 1)}
\]

lead term: \[
\frac{2x}{2x^3} = \frac{1}{x^2}
\]

Horizontal asymptote:

\[y = 0\]

Key values:

\[f(-1) = 4/5, \quad f(1) = 2/3, \quad f(7/2) = -4/49, \quad f(5) = 2/25\]

**Example 2:**

\[y = \frac{2x^3 - 8x^2}{2x - 6}\]

Reduce and factor:

\[
\frac{x^2(x-4)}{x-3}
\]

y-intercept: 0

x-intercepts: 0 (deg 2), 4 (deg 1)

Vertical asymptote:

\[
x = 3 \text{ (deg 1)}
\]

lead term:

\[
\frac{2x^3}{2x} = x^2
\]

Horizontal asymptote:

none

Key values:

\[f(-1) = 5/4, \quad f(1) = 3/2, \quad f(7/2) = -49/4, \quad f(5) = 25/2\]
Exponential functions

**Definition.** An exponential function is of the form \( y = b^x \) with the base \( b > 0 \).

\[ b^0 = 1, \quad b^1 = b, \quad b^2 = b \cdot b, \ldots \]

\[ b^{-n} = \frac{1}{b^n} \]

\[ b^{1/n} = \sqrt[n]{b}, \] the \( n \)th root of \( b \)

\[ b^{\rho q} = b^{(\rho/q)q} = (b^{1/q})^\rho \]

**Exponent Rules**

- \((b^a)^m = b^{am}\)
- \((5^2)^3 = (5^2)(5^2)(5^2) = 5^6\)
- \(b^n b^m = b^{n+m}\)
- \(5^2 5^3 = (5 \cdot 5)(5 \cdot 5 \cdot 5) = 5^5\)
- \(\frac{b^n}{b^m} = b^{n-m}\)
- \(\frac{5^3}{5^2} = 5^{-4} = \frac{1}{5^4} \cdot \frac{5^7}{5^3} = 5^4\)
- \((ab)^n = a^n b^n\)
- \(2^3 3^5 = 6^5\)
- \((\frac{a}{b})^n = a^n b^n\)
- \((\frac{2}{3})^5 = \frac{2^5}{3^5}\)

Simplify to an integer or a single exponent \(b^n\).

- \[ \sqrt{2^{\frac{1}{2}}} \sqrt{2^{\frac{3}{2}}} = \sqrt{2^{\frac{1}{2}+\frac{3}{2}}} = (2^{\frac{1}{2}})^4/2 = 2^{\frac{1}{4}/4} = 2^{1/4} = 2^{1/2} \]
- \(2^{\frac{3}{3}}) \sqrt{3} = 2^{\frac{3}{3}} \sqrt{3} = 2^3 = 8\)
- \[ \left(3^{2+\sqrt{5}}\right)\left(3^{2-\sqrt{5}}\right) = 3^{2+\sqrt{5}+2-\sqrt{5}} = 3^4 = 81\]
- \[ \frac{8^{1-n}}{8^{1/n}} = 8^{(1-n)-(1/n)} = 8^{1-1} = 8^{-2n} \]

**Property.** If \( b \neq 1 \), \( b^x = b^y \Leftrightarrow x = y \). Hence \( b^x \) is 1-1.

Gateway problems with the variable in the exponent: put both sides above a common base then equate exponents.

- \[ 27^x = 9 \]
  \[ (3^3)^x = 3^2 \]
  \[ 3 \cdot 3^x = 3^2 \quad \Rightarrow \quad 3^x = 2 \quad \Rightarrow \quad x = 2/3 \]

- \[ 8^{\frac{1}{x}} = 32/\sqrt{2} \]
  \[ 8^{\frac{1}{x}+1} = 2^5 2^{-1/2} \]
  \[ (2^3)^{x+1} = 2^{5-1/2} \]
  \[ 2^{3x+3} = 2^{9/2} \]
  \[ 2^{3x+3} = 2^{9/2} \]
  \[ 3x + 3 = 9/2 \]
  \[ 6x + 6 = 9 \]
  \[ 6x = 3 \]
  \[ x = 1/2 \]

- \[ z^8 = 32/\sqrt{2} \]
  \[ z^8 = 2^5/2^{1/2} = 2^{5-1/2} = 2^{\frac{9}{2}} \]
  \[ \therefore \quad z = 2^{\frac{9}{2}} \]

Note, this is different, the variable is in the base rather than the exponent. Raise both sides to the 1/8 power.

- \[ a \approx b \text{ means } a \text{ is approximately equal to } b. \]

**Fact:** \( 2^{10} = 10^3 \), i.e., \( 2^{10} \) is approximately equal to \( 10^3 \).

- Estimate \( 2^{0.8} \) and \( 2^{1.2} \)
  \[ 2^{0.8} = (2^{10})^{0.5} \approx (10^3)^{0.5} = 10^{1.5} \]
  \[ 2^{1.2} = (2^{10})^{1.2} \approx (10^3)^{1.2} = 10^6 \]

The graph of \( y = 1^x \) is the horizontal line \( y = 1 \).

Otherwise, the graph of \( y = b^x \)
- \( \cdot \) has y-intercept 1 but no x-intercept,
- \( \cdot \) it goes to \( \infty \) in one direction,
- \( \cdot \) it has the horizontal asymptote \( y = 0 \) in the other.

For \( b > 1 \), the graph of \( b^x \) is like the graph of \( 2^x \) as below. For \( 0 < b < 1 \), the graph is like \((b/2)^x\).

- Graph \( y = 2^{-x} \) Reflect \( 2^x \) across y-axis.
- Graph \( y = 2^{x-1} - 1 \) Shift right 1 unit, down 1 unit.
- Graph \( y = 2^{1-x} \) Shift left 1, reflect across y-axis.

**The function \( e^x \)**

**Definition.** \( e^x \) is the unique exponential function \( b^x \) whose the tangent at \((0, 1)\) has slope 1.

**Fact:** \( e \approx 2.7 \) Again, \( \approx \) means approximately equal.

Thus \( 2 < e < 3 \). Hence the graph of \( e^x \) lies between the graphs of \( 2^x \) and \( 3^x \).

Similarly, \( e^{-x} \) lies between \( 2^{-x} \) and \( 3^{-x} \).

- \[ y = e^{-x} \]
  \[ x \text{-intercept none} \]
  \[ y \text{-intercept 1} \]
  \[ \text{hor. asym. } y = 0 \]
  \[ \text{domain } (-\infty, \infty) \]
  \[ \text{range } (0, \infty) \]

- True or false?  
  **Recall:** \( e \approx 2.7 \)
  - \( \sqrt{e} < 1 \) **false** since \( 1 < e \Rightarrow \sqrt{1} < \sqrt{e} \Rightarrow 1 < \sqrt{e} \)
  - \( e^2 < 9 \) **true** since \( e < 3 \Rightarrow e^2 < 3^2 \Rightarrow e^2 < 9 \)
Math 140  Lecture 12

Exam 2 covers Lectures 7 - 12. Study the recommended exercises.
Review area, circumference, volume formulas - inside front cover.

RECALL. The graphs of $e^x$ and $e^{-x}$.

![Graph of $e^x$ and $e^{-x}$]

- x-intercept: none
- y-intercept: 1
- hor. asym.: y = 0
- vert. asym.: none
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- slope = 1
- log2
- $\log_2 y = \ln(x)$
- $x = e^{\log_2 y}$
- y = 1
- x = $\log_5 5^{\frac{3}{2}}$
- $\log_5 (y) = \log_8 (\sqrt{8})$
- $\log_8 2$
- $\log_8 2 = \log_8 \sqrt{8} = \log_8 8^{1/3} = 1/3$

Logarithms

Assume $b > 0$, $b \neq 1$. Thus $b^x$ is 1-1 and it has an inverse.

**DEFINITION.** $\log_b(x)$, the log of x to the base b, is the inverse of the exponential function $b^x$.

$\ln(x)$, the natural logarithm, $= \log_e(x)$ is the inverse of $e^x$.

Note, “ln” is “el-n” not “one-n” or “eye-n”.

Inverses act in opposite directions and inverses cancel. $	herefore$

- $y = \log_b(x)$ iff $b^y = x$.
- $y = \ln(x)$ iff $e^y = x$.

**FACT.** $e^0 = 1 \Rightarrow 0 = \ln(1)$.

- Simplify to a rational.
  - $\log_5 \sqrt{5}$
  - $\log_5 \sqrt{5} = \log_5 5^{1/2} = \log_5 5^{3/2} = \frac{3}{2}$.
  - $\log_8 \frac{1}{8}$
  - $\log_8 \frac{1}{8} = \log_8 \frac{1}{2} = \log_8 2^{-1} = −3$.
  - $\log_8 2$
  - $\log_8 2 = \log_8 \sqrt{8} = \log_8 8^{1/3} = 1/3$.

**Write in log form.** Either (method 1: easy problems) use the definition of log or (method 2: hard problems) take the appropriate log of both sides and cancel inverses.

- $2^3 = 8$
  - Method 1. Since 2^x and $\log_2(x)$ are inverse,
    - $3 = \log_2(8)$
  - Method 2. Take the log of both sides.
    - $\log_2(2^3) = \log_2(8)$ so $2x = \log_2(8)$

**Solve for x, write the answer using logarithms.**

- $5^{2x-1} = 6$
  - Method 1. Since 5^x and $\log_5(x)$ are inverse,
    - $3 = \log_5(6)$
  - Method 2. Take the log of both sides.
    - $\log_5(5^{2x-1}) = \log_5(6)$ so $2x − 1 = \log_5(6)$

**Logarithmic Functions**

**Graph $y = e^{x-1} - 1$.

Give the domain, range, intercepts and asymptotes.

![Graph of $y = e^{x-1} - 1$]

- x-intercept: $x = 1$
- y-intercept: $y = e^{-1} ≈ -0.63$
- hor. asym.: none
- vert. asym.: none
- domain: $(-\infty, \infty)$
- range: $(-1, \infty)$
- $e^{t+1} = 8$
  - $3t + 1 = \ln(8)$
  - $3t = \ln(8) − 1$
  - $t = \frac{1}{3}(\ln(8) − 1)$

**Write in log form.** Either (method 1: easy problems) use the definition of log or (method 2: hard problems) take the appropriate log of both sides and cancel inverses.

- $3^{2x} = 5^{x+1}$
  - Method 1. Since $3^x$ and $\log_3(x)$ are inverse,
    - $2x = \log_3(5)$
  - Method 2. Take the log of both sides.
    - $\log_3(3^{2x}) = \log_3(5^{x+1})$

**Solve for x.**

- $2x \ln 3 = (x + 1) \ln 5$
- $2x \ln 3 = \ln(5^{x+1})$
- $2 \ln 3 - \ln 5 = \ln 5$
- $x = \ln 5/(2 \ln 3 - \ln 5)$

Since they are inverses, the graph of $\log_b(x)$ is the reflection of $b^x$ across the major diagonal and $\ln(x)$ is the reflection of $e^x$.

**Graph $y = \ln(x)$**

For the graph of $\ln(x)$

- x-intercept: 1
- y-intercept: none
- vert. asym.: $x = 0$
- hor. asym.: none
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$.

**Graph $y = \ln(x + 1) + 1$**

- y-intercept: $\ln(0+1) + 1 = 1$
- x-intercept: $\ln(x+1) + 1 = 1$
- $\ln(x+1) = -1$
- $x + 1 = e^{-1}$
- $x = e^{-1} − 1$

**Simplify to a rational.**

- $\ln(0+1) + 1 = 1$
- $\ln(x+1) + 1 = 0$
- $\ln(x+1) = -1$
- $x + 1 = e^{-1}$
- $x = 1/e − 1 = 1/(2.7) − 1 = -0.63$

**Graph $y = 1 − \ln(1−x)$**

Left 1, refl. across y, refl. across x, up 1.
Recall: Since \( \log_b x \) and \( b^x \) are inverses of each other:

- \( \log_b x = y \iff x = b^y \)
- \( \log_b b^x = x \)
- \( b^{\log_b x} = x \)

Most properties of logarithms are the same as for exponentiation but with

<table>
<thead>
<tr>
<th>the symbols</th>
<th>0</th>
<th>+</th>
<th>−</th>
<th>((−)^n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>exchanged with</td>
<td>1</td>
<td>×</td>
<td>÷</td>
<td></td>
</tr>
</tbody>
</table>

Assume \( b > 0, b \neq 1 \). On the log side, assume \( x, y > 0 \).

### LOG PROPERTIES          EXONENT PROPERTIES

- \( \log_b 1 = 0 \)
- \( \log_b b = 1 \)
- \( \log_{b^t} x = \frac{\log_b x}{t} \)
- \( \log_{b^t} (b^t x) = \log_b x + \log_b x \) iff \( \log_b x \) is defined.
- \( \log_{b^t} x - \log_{b^t} y = \log_{b^t} (\frac{x}{y}) \) iff \( \log_b x \) and \( \log_b y \) are defined.
- \( \log_{b^t} x + \log_{b^t} y = \log_{b^t} (xy) \) iff \( \log_b x \) and \( \log_b y \) are defined.

Likewise for \( \log_{b^{-t}} x = \log_b x - \log_b y \).

- \( \log_b x^n = n \log_b x \) iff \( \log_b x \) is defined.
- \( \log_{b^t} x^n = b^{n \log_b x} \) iff \( \log_b x \) is defined.
- \( x^t = \{b^{\log_b x}\}^n \) iff \( \log_b x \) is defined.
- \( x^t = x^n \) iff \( \log_b x \) is defined.

When \( b = e \), \( \log_{e^t} (e) = 1 \) becomes: \( \ln(e) = 1 \)

Note: \( \log_b (x \cdot y) = \log_b x + \log_b y \neq \log_b (x + y) \). The last term cannot be broken into simpler pieces.

Simplify.

- \( \ln e^2 - \ln e^4 + \ln 1 - \ln e = 2 - 4 + 0 - 1 = -3 \)
- \( e^{\ln 2} - e^{\ln 4} + e^{\ln 1} - e^{\ln e} = 2 - 4 + 1 - e = -1 - e \)

Combine into a single logarithm.

- \( 2 \log_{10} x + \log_{10} y = \log_{10} x^2 + \log_{10} y = \log_{10} (x^2y) \)
- \( \log_{10} x - 4 \log_{10} y = \log_{10} x - \log_{10} y^4 = \log_{10} \left( \frac{x}{y^4} \right) \)
- \( \ln (x^2 - y^2) - \ln (y^2 + 1) = \ln \frac{x^2 - y^2}{y^2 + 1} \)

### Change of base formula

\( \log_a x = \frac{\log_b x}{\log_b a} \).

**Proof:** \( \log_{b^t} x = \log_{b^t} (a^{\log_{b^t} x}) = \log_{b^t} (x)\log_{b^t} a \) now divide both sides by \( \log_{b^t} a \).

- Express in terms of \( \log_{10} \): \( \log_2 2 = \frac{\log_{10} 2}{\log_{10} 2} \)
- Express in terms of natural logarithms \( \ln \): \( \log_5 t = \frac{\ln t}{\ln 5} \)
Math 140   Lecture 14

Exponential growth
A bacteria colony starts with 10 bugs. Each bug splits into two bugs every hour. How many bugs are there after \( t \) hours?

<table>
<thead>
<tr>
<th>Number of hours</th>
<th>Number of bugs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>10(^2)</td>
</tr>
<tr>
<td>2</td>
<td>(10(^2))(^2) = 10(^4)</td>
</tr>
<tr>
<td>3</td>
<td>(10(^2))(^3) = 10(^6)</td>
</tr>
<tr>
<td>( t ) hours</td>
<td>10(^t)</td>
</tr>
</tbody>
</table>

Exponential functions measure the size of a growing population, the amount of money in a compound interest account, the number of atoms left after radioactive decay, etc. \( N(t) \) = the amount at time \( t \).

**BASE-\(e\) FORM LEMMA.** Every exponential function can be written in the form \( N(t) = N_0e^{kt} \).

- \( N_0 \) = the initial amount.
- \( k \), the coefficient of \( t \), is the growth constant.
- If \( k > 0 \), \( N(t) \) measures exponential growth.
- If \( k < 0 \), \( N(t) \) measures exponential decay.

**FACT:** Every \( a \) > 0 is a power of \( e \): \( a = e^{\ln a} \).

Example: \( 2 = e^{\ln 2} \).

- Write the bug population \( N(t) = 10 \cdot 2^t \) in base-\( e \) form. \( N(t) = 10 \cdot 2^t = 10 \cdot (e^{\ln 2})^t = 10 \cdot e^{(\ln 2)t} \).

Hence, in base-\( e \) form, \( N(t) = 10e^{(\ln 2)t} \).

Thus \( N_0 = 10 \), the initial amount and \( k = \ln(2) \), the natural log of the initial base.

- You put \$4,000 in an account at 5% interest compounded annually. Write the amount \( N(t) \) of money in the account after \( t \) years in base-\( e \) form.

<table>
<thead>
<tr>
<th>Number of years</th>
<th>Amount ( N(t) ) in account</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4,000</td>
</tr>
<tr>
<td>1</td>
<td>4,000+(0.05)4,000 = 4,000(1.05)</td>
</tr>
<tr>
<td>2</td>
<td>4,000(1.05)+(0.05)4,000(1.05) = 4,000(1.05)(1+0.05) = 4,000(1.05)^2</td>
</tr>
<tr>
<td>3</td>
<td>4,000(1.05)^3</td>
</tr>
<tr>
<td>( t ) years</td>
<td>4,000(1.05)^t</td>
</tr>
</tbody>
</table>

Hence \( N(t) = 4000(1.05)^t = 4000(e^{(\ln 1.05)t}) = 4000e^{(\ln 1.05)t} \).

\( \therefore N(t) = 4000e^{(\ln 1.05)t} \).

\( \therefore N_0 = 4000 \) and \( k = \ln(1.05) \).

In each problem, first write \( N(t) \) in base-\( e \) form, then solve.

- A bacteria colony starts with \( 10^3 \) bugs. Four hours later it has \( 5 \times 10^3 \) bugs.

(a) Find the growth constant \( k \).

(b) What is the population two hours after the start?

(c) How long will it take for the population to triple?

First find \( N(t) = \) the number of bugs after \( t \) hours.

\( N_0 = 10^3 \)

\( N(4) = 5 \times 10^3 \)

\( N(t) = N_0e^{kt} \)

\( \therefore N(t) = 10^3e^{kt} \)

\( 10^3e^{4k} = N(4) = 5 \times 10^3 \)

\( e^{4k} = 5 \)

\( \ln(e^{4k}) = \ln 5 \quad \leftarrow \text{may omit this alternate step} \)

\( 4k = \ln 5 \)

\( k = \frac{\ln 5}{4} \)

\( N(t) = 10^3e^{\frac{\ln 5}{4}t} \quad \leftarrow \text{base-} e \text{ form} \)

(a) The growth constant is \( k = \frac{\ln 5}{4} \).

(b) After two hours the population is

\( N(2) = 10^3e^{\frac{\ln 5}{4}} = 10^3e^{0.693} \)

(c) Let \( t \) be the time when the population has tripled.

\( \therefore N(t) = 3 \times \) the initial amount \( N_0 \)

\( \therefore N(t) = 3 \cdot 10^3 \)

\( \therefore 10^3e^{\frac{\ln 5}{4}t} = 3 \cdot 10^3 \)

\( \therefore e^{\frac{\ln 5}{4}t} = 3 \)

\( \therefore \frac{\ln 5}{4}t = \ln 3 \)

\( \therefore t = \frac{\ln 3}{\ln 5} = \frac{\ln 3}{\ln 10/2} = \frac{\ln 3}{\ln 2^2} = \frac{\ln 3}{2\ln 2} = \frac{1}{2}\ln 3 \)

The population triples after \( \frac{\ln 3}{\ln 5} \) hours.

- Initially a sample has \$8 \) lbs of a radioactive substance with a half-life of 5 days, i.e., half decays after 5 days. How many lbs remain after 3 days?

First find \( N(t) \) = the number of lbs left after \( t \) days.

Given:

\( N_0 = 8 \)

\( N(5) = 4 \)

\( N(t) = N_0e^{kt} \)

\( \therefore N(t) = 8e^{kt} \)

\( 8e^{5k} = N(5) = 4 \)

\( e^{5k} = \frac{1}{2} \)

\( 5k = \ln \frac{1}{2} \)

\( k = \frac{1}{5} \ln(1/2) \)

\( N(t) = 8e^{\frac{1}{5}\ln(1/2)t} \quad \leftarrow \text{base } e \text{ form} \)

\( N(3) = 8e^{\frac{1}{5}\ln(1/2)3} = 8e^{\frac{3}{5}\ln 1/2} \)

\( \therefore 8e^{\frac{3}{5}\ln 1/2} \) lbs remain after 3 days.

This is a decay since \( \ln(1/2) = \ln(2^{-1}) = -\ln(2) \) is negative.
Math 140  Lecture 15

Angles, arcs, and radians

RECALL. A circle of radius \( r \) has circumference \( 2\pi r \).
\( \pi \approx 3.14 \). Unit circle circumference = \( 2\pi r = 2\pi (1) = 2\pi \).
\( \theta \) and \( \omega \) are the Greek letters “theta” and “omega”.

DEFINITION. Suppose the vertex of an angle is at the center of a circle of radius \( r \). Let \( s \) be the length of the arc the angle intercepts on the circle. Then
\[
\theta = \frac{s}{r}
\]
is the \textit{radian measure} of the angle. For unit circles, the radius \( r = 1 \) and radian measure equals arc length: \( \theta = s \).

- Radian and degree measures on the unit circle.

\[\begin{array}{c}
\pi/2 = 90^\circ \\
\pi/4 = 45^\circ \\
180^\circ = \pi \\
0^\circ = 0, \quad 360^\circ = 2\pi \\
-\pi/4 = -45^\circ
\end{array}\]

Clockwise angles \( \nearrow \) are negative.

CONVERSION FORMULAS. \( 180^\circ = \pi \) radians. Thus
\( 1^\circ = \pi/180 \) radians; \( 1 \) radian = \( 180/\pi \) \ degree.

- Convert 100° to radians.
\[100^\circ = 100 \cdot \frac{\pi}{180} \approx 1.74 \text{ radians}\]
- Convert \( \pi/6 \) radians to degrees.
\[\frac{\pi}{6} \text{ radians} = \frac{\pi}{6} \cdot \frac{180}{\pi} = 30^\circ .\]

When \( \theta \) is in radians, we can solve \( \theta = s/r \) for arclength:
\[s = r \theta \]

- Find the length of a 30° arc on a circle of radius 12 inches.
First convert to radians. By the above 30° = \( \pi/6 \) radians.
\[\therefore s = r \theta = \frac{\pi}{6} \cdot 12 = 2\pi \text{ inches} .\]

- Find the radian measure of an angle which intercepts a 5 inch arc on a circle of radius 12 inches.
\[\theta = \frac{s}{r} = \frac{5}{12} \text{ radians}\]

Speed

DEFINITION. If an object travels a distance \( d \) in time \( t \), its \textit{linear speed} is \( \frac{d}{t} \).

- If an object rotates through an angle \( \theta \) in time \( t \), its \textit{rotational speed} is \( \omega = \frac{\theta}{t} \).

THEOREM. If a point rotates around a circle of radius \( r \) with rotational speed \( \omega \), then its linear speed is \( \omega r \).

Proof. If a point rotates through an angle \( \theta \) on a circle of radius \( r \) in time \( t \), then \( \omega = \frac{\theta}{t} \) and the distance \( d \) it travels = the length of the arc it traces = \( \theta r \). \( \therefore \) its linear speed = \( \frac{d}{t} = \theta r/t = (\theta/t)r = \omega r \).

- A point revolves around a circle of radius 3 feet at 10 revolutions per minute.
(a) What is its rotational speed (in radians)?
\[1 \text{ revolution} = 2\pi \text{ radians}, \therefore \]
\[\omega = 10 \text{ revs/min} = 10 \cdot 2\pi \text{ radians/min} = 20\pi \text{ radians/min} .\]
(b) What is its linear speed? Its linear speed
\[= \omega r = (20\pi \text{ rad/min}) \cdot (3 \text{ feet}) = 60\pi \text{ feet/min} .\]

Radian and degree measures on the unit circle.

\( 3.14 \). Unit circle circumference = \( 2\pi r = 2\pi (1) = 2\pi \).

The six trigonometric functions of \( \theta \) are:
\[
\begin{align*}
\sin \theta &= y \\
\cos \theta &= x \\
\tan \theta &= \sin \theta / \cos \theta \\
\cot \theta &= \cos \theta / \sin \theta \\
\sec \theta &= 1 / \cos \theta \\
\csc \theta &= 1 / \sin \theta
\end{align*}
\]

- Draw an angle of \( \pi/2 \) radians in standard position. Find the six trigonometric functions.
\[\sin(\pi/2) = 1 \quad \csc(\pi/2) = 1 \]
\[\cos(\pi/2) = 0 \quad \sec(\pi/2) = \text{undef} \]
\[\tan(\pi/2) = \text{undef} \quad \cot(\pi/2) = 0 \]

- A point \((x, y)\) on the unit circle is in the second quadrant and \( y = \frac{3}{4} \). Find the six trig functions for \( \theta \).
\[\begin{align*}
\theta &= \frac{3}{4} \quad \text{First, find } x. \\
(x, y) \text{ on the unit circle} &\Rightarrow x^2 + y^2 = 1. \\
\therefore x^2 = 1 - y^2 = 1 - \left(\frac{3}{4}\right)^2 = 1 - \frac{9}{16} = \frac{7}{16}.
\end{align*}\]
\[x = \pm \sqrt{7}/4 \quad \text{in the second quadrant } \Rightarrow x \text{ is negative}. \\
\therefore x = -\sqrt{7}/4.
\]
\[\begin{align*}
\sin \theta &= 3/4 \\
\cos \theta &= -\sqrt{7}/4 \\
\tan \theta &= -3/\sqrt{7}
\end{align*}\]
Math 140  Lecture 16
Fri. = last day to withdraw. Keller 419A secretary will sign for me.

FACTS. Know the sin and cos (and hence tan, cot, sec, csc) of:
\[0, \pi/6, \pi/4, \pi/3, \pi/2\].

DEFINITION. For any angle in standard position (vertex = the origin, initial side = the positive x-axis), its reference angle is the acute positive angle (thus in [0, \pi/2]) between the x-axis (positive or negative, whichever is nearest) and the angle’s terminal side.

THEOREM. The sin and cos of \(\theta\) equals the sin and cos of its reference angle except for the sign which is determined by \(\theta\)’s quadrant.

1. **Rewrite in terms of reference angles, then evaluate.**
   - \(\cos(120^\circ) = -\cos(60^\circ) = -1/2\) quadrant II
   - \(\sin(120^\circ) = \sin(60^\circ) = \sqrt{3}/2\) quadrant II
   - \(\cos(3\pi/4) = -\cos(\pi/4) = -1/\sqrt{2}\) quadrant II
   - \(\sin(3\pi/4) = \sin(\pi/4) = 1/\sqrt{2}\) quadrant II
   - \(\cos(-3\pi/4) = -\cos(\pi/4) = -1/\sqrt{2}\) quadrant III
   - \(\sin(-3\pi/4) = -\sin(\pi/4) = -1/\sqrt{2}\) quadrant III

2. **List three angles (in radian measure) whose cos is \(1/2\).**
   - \(\pi/3, -\pi/3, 5\pi/3\)

NOTATION. \(\sin\theta\) means \(\sin(\theta)\); \(\sin^2\theta\) means \((\sin(\theta))^2\).

THEOREM. Since \(\sin\theta\) and \(\cos\theta\) are the legs of a right triangle of hypotenuse 1,
\[\sin^2\theta + \cos^2\theta = 1\].

- \(\sin\theta = -2/3\) and \(\pi < \theta < 3\pi/2\). Find \(\cos\theta\) and \(\tan\theta\).
  - \((-2/3)^2 + \cos^2\theta = 1\)
  - \(\cos\theta = 1 - 4/9 = 5/9\)
  - \(\cos\theta = \pm \sqrt{5}/3\)
  - \(\pi < \theta < 3\pi/2\) is in quadrant III, hence \(\cos\theta < 0\).
  - \(\cos\theta = -\sqrt{5}/3\)
  - \(\tan\theta = \sin\theta/\cos\theta = 2/\sqrt{5}\)

- \(\sec\theta = -3\) and \(\sin\theta < 0\). Find \(\tan\theta\). First find \(\sin, \cos\).
  - \(\cos\theta = -1/3\)
  - \(\sin^2\theta + (-1/3)^2 = 1\)
  - \(\sin^2\theta = 1 - 1/9 = 8/9\)
  - \(\sin\theta = \pm \sqrt{8}/3\). Since \(\sin\theta < 0\),
  - \(\sin\theta = -\sqrt{8}/3 = -2\sqrt{2}/3\).
  - \(\tan\theta = 2\sqrt{2}\).

- **Simplify** \((\cos \theta + 1)(\tan \theta + \sin \theta)\)
  - \(= \cos \theta \tan \theta + \cos \theta \sin \theta + \tan \theta + \sin \theta\)
  - \(= \tan \theta \sin \theta + \cos \theta \sin \theta + \tan \theta + \sin \theta\)
  - \(= 2 \sin \theta + \cos \theta \sin \theta + \tan \theta\).

Note \(y^2 + 3y + 2\) factors into \((y+1)(y+2)\).

- **Factor**:
  - \(\csc^2 \theta + 3 \csc \theta + 2 = (\csc \theta + 1)(\csc \theta + 2)\).

Note \(x^2 - y^2 = (x-y)(x+y)\).

- **Factor**:
  - \(\sin^2 B - \cos^2 B = (\sin B - \cos B)(\sin B + \cos B)\).
RECALL. \( \sin^2 \theta + \cos^2 \theta = 1 \)

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} \\
\sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta}
\]

**Simplify.**

- \( \cot \theta \sec \theta \cos \theta = \frac{\cos \theta}{\sin \theta \cos \theta} \cdot \cos \theta = \cot \theta \)
- \( \frac{\sin x - 1}{\sin x} + 1 \cdot \frac{\sin x + 1 - \sin x}{\sin x} = \frac{2 \sin x - 1}{1} = 2 \sin x - 1 \)

**DEFINITION.** An **identity** is an equation which is always true.

- \( x^2 + y^2 = 1 \), \( x, y = 2x \) aren’t identities: they aren’t always true. \( \sin^2 \theta + \cos^2 \theta = 1 \), \( xy = yx \) are identities, i.e., always true.

To prove an identity either (1) start from one side (usually the more complicated side) and work your way to the other side or (2) show that the equality is equivalent to a truth. The text uses (1) we’ll give examples of (2).

- **Prove** \( \cot^2 \theta = \csc^2 \theta - 1 \)
  
  \[ \text{iff} \quad \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} - 1 \quad \text{iff} \quad \cos^2 \theta = 1 - \sin^2 \theta \]

- **Prove** \( \cos \tan \theta = \sec \theta \sin \theta \) (by cross multiplying)

**Right triangles**

The trig functions were defined on the unit circle where the hypotenuse of the right triangle had length 1. For the angle pictured, \( \sin \theta = y \). Any other right triangle with an angle \( \theta \) is similar to this unit-circle triangle. Hence the ratios of the corresponding sides are equal. Thus \( \sin \theta = y = y/1 = \text{alc} = \text{side opposite/hypotenuse} \).

**THEOREM.** For any right triangle with an acute angle \( \theta \):

- \( \sin \theta = \text{opposite/hypotenuse}^* \)
- \( \cos \theta = \text{adjacent/hypotenuse} \)
- \( \tan \theta = \text{opposite/adjacent} \)

* More precisely, “length of the side opposite/length of the hypotenuse”. We’ll abbreviate this to just: opp/hyp.

In a right triangle, the two complementary small angles add up to 90°. If one is 0, the other is 90°-θ or \( \pi/2-\theta \).

**THEOREM.**

\[
\sin(\frac{\pi}{2} - \theta) = \cos \theta \quad \sin(90^\circ - \theta) = \cos \theta \\
\cos(\frac{\pi}{2} - \theta) = \sin \theta \quad \cos(90^\circ - \theta) = \sin \theta
\]

**Proof.** In the picture below,

\[
\sin \theta \quad \cos(90^\circ - \theta) \quad \cos \theta \quad \sin(90^\circ - \theta)
\]

\[
\text{sin} \theta \quad \text{cos}(90^\circ - \theta) \quad \text{cos} \theta \quad \text{sin}(90^\circ - \theta)
\]

In Fig. 1, find sin, cos and tan for \( \theta \) and \( 90^\circ - \theta \).

First find the third side.

- The hypotenuse = \( \sqrt{5^2 + 6^2} = \sqrt{61} \)
- \( \sin \theta = 6/\sqrt{61} \)
- \( \cos \theta = 5/\sqrt{61} \)
- \( \tan \theta = 6/5 \)
- \( \sin(90^\circ - \theta) = 5/\sqrt{61} \)
- \( \cos(90^\circ - \theta) = 6/\sqrt{61} \)
- \( \tan(90^\circ - \theta) = 5/6 \)

In Fig. 2, find \( \sin B \), \( \cos B \) and \( \tan B \).

Let \( y \) be the third side. Then \( y^2 + (x-1)^2 = (9x^2)^2 \)

- \( y^2 = 81x^4 - (x-1)^2 = 81x^4 - (x^2 - 2x + 1) \)
- \( y^2 = 81x^4 - x^2 + 2x - 1 \)
- \( y = \sqrt{81x^4 - x^2 + 2x - 1} \)

- \( \sin B = y/9x^2 = \sqrt{81x^4 - x^2 + 2x - 1} / 9x^2 \)
- \( \cos B = (x-1)/(9x^2) \)
- \( \tan B = y(x-1) = \sqrt{81x^4 - x^2 + 2x - 1} / (x-1) \)

- \( \tan \theta = 2/3 \), find \( \sin \theta \) and \( \cos \theta \), where \( \theta \) is acute.

Since \( \tan \theta = \text{opposite/adjacent} = 2/3 \), draw a right triangle with opposite = 2 and adjacent = 3 (the drawing need not be exact). Then find the third side.

- \( y = \sqrt{3^2 + 2^2} = \sqrt{13} \)
- \( \sin \theta = 2/\sqrt{13} \)
- \( \cos \theta = 3/\sqrt{13} \)
Math 140    Lecture 18
Exam 3 covers lectures 12 -18. Study the recommended exercises.

**Definition.** For any function \( f \): \( f \) is even iff \( f(-x) = f(x) \).
\( f \) is odd iff \( f(-x) = -f(x) \).

Graphically, the left half (the half plane left of the \( y \)-axis) of an even function is of the right half across the \( y \)-axis. The left half of an odd function is the negative of this reflection.

- \( f(x) = x^2 \) is even since \( f(-x) = (-x)^2 = x^2 = f(x) \).
- \( g(x) = x^3 \) is odd since \( g(-x) = (-x)^3 = -x^3 = -g(x) \).

**Figure:**

\[ x^2 \] \hspace{1cm} \[ x^3 \]

\( \text{even} \) \hspace{1cm} \( \text{odd} \)

Instead of thinking of \( \sin \theta \) as a function of an angle \( \theta \), we can think of it as a function \( \sin(t) \) of a real variable \( t \).

**Minus Theorem.** \( \sin(-t) = -\sin(t) \), \( \cos(-t) = \cos(t) \), \( \tan(-t) = -\tan(t) \). Thus \( \cos(t) \) is even; \( \sin(t) \) and \( \tan(t) \) are odd functions.

**Proof.**

\[ \tan(-t) = \tan(t)/\cos(t) = -\sin(t)/\cos(t) = -\tan(t). \]

**Definition.** A function \( f \) is periodic with period \( p \) iff \( f(x+p) = f(x) \)
for all \( x \). \( p \) must be the smallest such positive number.

**2π Theorem.** \( \sin(t+2\pi) = \sin(t) \), \( \cos(t+2\pi) = \cos(t) \), for integers \( k \), \( \sin(t+2k\pi) = \sin(t) \), \( \cos(t+2k\pi) = \cos(t) \).

**Proof.** See the picture for \( \sin \) and \( \cos \). \( \tan(t+\pi) = \sin(t+\pi)/\cos(t+\pi) = -\sin(t)/-\cos(t) = \sin(t)/\cos(t) = \tan(t) \).

Thus \( \sin \) and \( \cos \) have period \( 2\pi \), but \( \tan \) has period \( \pi \).

**Simplify:** \( \cos(3\pi - x) \).
\[ \cos(3\pi - x) = \cos(2\pi + \pi - x) = \cos(\pi - x) = -\cos(-x) = -\cos(x). \]

Previously we found reference angles in \([0, \pi/2]\) geometrically. Now we can find them algebraically.

Rewrite in terms of the nearest multiple of \( \pi \). Throw out this multiple of \( \pi \) using the \( 2\pi \) or \( \pi \) Theorem. Handle minus signs using the Minus Theorem.

**Rewrite with a reference angle in \([0, \pi/2]\), then evaluate.**

- \( \sin(5\pi/6) \), \( 5\pi/6 \) is near \( 6\pi/6 = \pi \).
- \( \sin(5\pi/6) = \sin(\pi - \pi/6) = -\sin(\pi/6) = \sin(\pi/6) = 1/2. \)
- \( \cos(4\pi/3) \), \( 4\pi/3 \) is near \( 3\pi/3 = \pi \).
- \( \cos(4\pi/3) = \cos(3\pi/3 + \pi/3) = \cos(\pi/3) = -\cos(\pi/3) = -1/2. \)

**Pythagorean Identities.** \( \sin^2 t + \cos^2 t = 1 \), \( \tan^2 t + 1 = \sec^2 t \), \( \cot^2 t + 1 = \csc^2 t \).

**Proof.** We’ve proved the first. To prove the second, multiply both sides by \( \cos^2 t \). For the third, multiply both sides by \( \sin^2 t \).

## Simplify
\[ \csc t + \csc t \cot^2 t = \sec^2 t - \tan^2 t \]
\[ = \csc t [1 + \cot^2 t] \]
\[ = \csc t \csc^2 t \]
\[ = \csc^3 t \]

**Simplify**
\[ \frac{\sin^2 t - \cos^2 t}{\sin^4 t - \cos^4 t} = 1 \]

**Prove** \( \cot \theta + \tan \theta + 1 = \frac{\cot \theta}{1 - \tan \theta} + \frac{\tan \theta}{1 - \cot \theta} \)

Let \( x = \tan \theta \), then \( \cot \theta = 1/\tan \theta = 1/x \).

Thus we must prove
\[ \frac{1}{x} + x + 1 = \frac{1/x}{1-x} + \frac{x}{1-1/x} \]

**Simplify**
\[ \frac{1}{x} + x + 1 = \frac{1/x}{1-x} - \frac{x^2}{1-x} \]

**iff** \( 1 + x^2 + x = [1 - x^3]/[1-x] \)

**iff** \( (x^2 + x + 1)(1-x) = (1-x^3) \)

**iff** \( 1 - x^3 = 1 - x^3 \)

**iff** true
Math 140 Lecture 19

Graphs of sin and cos

**Recall.** sin and cos have period $2\pi$:

$$\sin(x+2\pi) = \sin(x), \cos(x+2\pi) = \cos(x).$$

We often graph periodic functions only over one period, e.g., $[0, 2\pi]$. Before and after this interval, they repeat.

Graph of $\sin(x)$.

At $x=0$, the line $y=x$ is tangent to the graph of $\sin(x)$.

Graph of $\cos(x)$.

Done similarly ... .

**Definition.** The amplitude of a function $f$ is half the difference between the max and min values of $f$.

Like periods, amplitudes are always positive.

- Find the amplitude and period of $f$, $g$, and $h$.

- Graph $y=2\sin(x)$ over one period.

- Graph $y=-3\cos(\pi x)$ over one period.

For $A>0$, $y=\pm \text{Asin}(x)$ has amplitude $A$. Reflect the graph across the x-axis for $-\text{Asin}(x)$.

---

**Theorem.** For $y=\pm \text{Asin}(Bx)$ & $y=\pm \text{Acos}(Bx)$ with $A,B>0$,

- amplitude: $A$
- period: $p=2\pi/B$

Graph $y=-3\cos(\pi x)$ over one period. List the amplitude, period, x-intercepts and the intervals in the period on which the function increases.

$y=-3\cos(\pi x) = -\text{Acos}(Bx)$

$A=3, B=\pi$
- amplitude: $A=3$
- period: $p=2\pi/B=2\pi/\pi=2$
- x-intercepts?
  - $\cos(x)=0$ when $x= ..., \pi/2, 3\pi/2, ...$
  - $-3\cos(\pi x)=0$ when $\pi x= ..., \pi/2, 3\pi/2, ...

  - $\text{iff } x= ..., 1/2, 3/2, ...$

increases on: $[0,1]$.

- Find an equation for the graph. Write it in the form $y=\pm \text{Asin}(Bx)$ or $y=\pm \text{Acos}(Bx)$ with $A,B>0$.

The graph has the shape of $y=-\text{Acos}(Bx)$.

- amplitude: $A=2$; period: $p=4$
- $B: p=2\pi/B \Rightarrow 4=2\pi/B \Rightarrow 4B=2\pi \Rightarrow B=\pi/2$.

Equation: $y=-2\cos(\pi x)$
RECALL. For \( C > 0 \), the graph of 
f(x+C) is the graph of f(x) shifted \( C \) units to the left.  
f(x−C) is the graph of f(x) shifted \( C \) units to the right.  
The phase shift is the amount \( C \) subtracted from \( x \).  
f(x+C) = f(x−(−C)) has phase shift \(-C\).  
f(2x−C) = f(2(x−C/2)) has phase shift \( C/2 \).  

- Graph \( y = \sin(x−\pi/2) \).  
  Phase shift = \( \pi/2 \).

To graph \( y = A \sin(Bx \pm D) \):
- Factor out \( B \), and rewrite in the form \( A \sin(B(x-D)) \):
  \[ A \sin(Bx \pm D) \rightarrow A \sin(B(x \pm (D/B))) \rightarrow A \sin(B(x-D)) \].
- Graph \( y = \sin(Bx) \), stretch vertically to get \( y = A \sin(Bx) \).
- Shift horizontally to get \( y = A \sin(B(x-D)) \).

Graph \( y = \sin(2(x−\pi/2)) \) over one period.
\[ \sin(2x) = \sin(2(x−\pi/2)) \]  
Phase shift = \( \pi/2 \)

\[ \pi \] is not the shift for \( \sin(2x) \). In \( \sin(2x) \), \( \pi \) is subtracted from \( 2x \).  
But the shift must be subtracted from \( x \). You must factor \( \sin(2x-\pi) \) to \( \sin(2(x-\pi/2)) \) to see that the shift subtracted from \( x \) is \( \pi/2 \).

Graph \( -2 \cos(2x−\pi/2) \) over one period.
\[ -2 \cos(2x−\pi/2) = -2 \cos(2(x−\pi/4)) \]  
\[ \therefore \text{amplitude} = 2, \text{period} = \pi, \text{phase shift} = \pi/4. \]

Graph \( y = \csc(x) \).
\[ \csc(x) = 1/\sin(x) \]

To graph \( \csc(x) \), graph \( \sin(x) \) and then invert its values.
\[ \sin(x) = \frac{1}{2} \Rightarrow \csc(x) = 1/(\frac{1}{2}) = 2. \]
\[ \sin(x) = 0 \Rightarrow \csc(x) = 1/0 = \text{undef.} \]

Graph \( y = -\csc(3x+\pi) \) over one period.
**Math 140  Lecture 21**

**RECALL.** \( \sin^2 x + \cos^2 x = 1, \) \( \sin(-x) = -\sin(x), \) \( \cos(-x) = \cos(x). \)
\( \cos(\pi/2 - x) = \sin(x), \) \( \sin(\pi/2 - x) = \cos(x). \)

**TRIGONOMETRIC ADDITION FORMULAS.**

\[
\begin{align*}
\sin(s \pm t) &= \sin s \cos t \pm \cos s \sin t \\
\cos(s + t) &= \cos s \cos t - \sin s \sin t \\
\cos(s - t) &= \cos s \cos t + \sin s \sin t \\
\tan(s + t) &= \frac{\tan s + \tan t}{1 - \tan s \tan t} \\
\tan(s - t) &= \frac{\tan s - \tan t}{1 + \tan s \tan t}
\end{align*}
\]

**Proof of** \( \cos(s + t) = \cos s \cos t - \sin s \sin t. \)

**In each problem below and in homework:**

1. Apply an addition formula. first point
2. Simplify. second point

\[
\begin{align*}
cos \frac{\pi}{5} \cos \frac{6\pi}{30} + \sin \frac{\pi}{5} \sin \frac{3\pi}{30} \\
= \cos \left( \frac{\pi}{5} - \frac{\pi}{30} \right) \\
= \cos \left( \frac{6\pi}{30} - \frac{\pi}{30} \right) = \cos \left( \frac{\pi}{6} \right) = \sqrt{3}/2
\end{align*}
\]

\[
\begin{align*}
sin(x + y) \cos y - \cos(x + y) \sin y \\
= \sin((x + y) - y) = \sin x
\end{align*}
\]

\[
\begin{align*}
sin(a + b) + \sin(a - b) \\
= (\sin a \cos b + \cos a \sin b) + (\sin a \cos b - \cos a \sin b) \\
= 2 \sin a \cos b
\end{align*}
\]

\[
\begin{align*}
\cos(a - b) - \cos(a + b) \\
= (\cos a \cos b + \sin a \sin b) - (\cos a \cos b - \sin a \sin b) \\
= 2 \sin a \sin b
\end{align*}
\]

\[
\begin{align*}
\tan \frac{\pi}{3} - \tan \frac{\pi}{6} \\
= \frac{\tan \left( \frac{\pi}{3} - \frac{\pi}{6} \right)}{1 + \tan \frac{\pi}{3} \tan \frac{\pi}{6}} \\
= \tan \left( \frac{\pi}{6} \right) = \frac{\sin(\pi/6)}{\cos(\pi/6)} \\
= \frac{\sqrt{3}/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
2 \tan \left( \frac{\pi}{8} \right) \\
= \frac{\sin \left( \frac{\pi}{8} + \frac{\pi}{8} \right)}{1 - \tan^2 \left( \frac{\pi}{8} \right)} \\
= \frac{\tan \left( \frac{\pi}{4} \right)}{1 - 2 \tan^2 \left( \frac{\pi}{8} \right)} = \tan \left( \frac{\pi}{4} \right) = 1
\end{align*}
\]

**Given** \( \tan s = \frac{1}{3}, \) \( \tan t = \frac{2}{3}, \) **find** \( \tan(s + t). \)

\[
\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t} \\
= \frac{\frac{1}{3} + \frac{2}{3}}{1 - \frac{1}{3} \cdot \frac{2}{3}} = \frac{1}{1 - \frac{2}{9}} = \frac{9}{7}
\]

**Use an addition formula to find** \( \tan(\pi/12). \)

\( \pi/12 = 4\pi/12 - 3\pi/12 = \pi/3 - \pi/4 \)

\[
\begin{align*}
\tan \left( \frac{\pi}{12} \right) &= \tan \left( \frac{\pi}{3} - \frac{\pi}{4} \right) \\
= \frac{\tan(\pi/3) - \tan(\pi/4)}{1 + \tan(\pi/3) \tan(\pi/4)} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}} = \frac{(\sqrt{3} - 1)(1 - \sqrt{3})}{(1 + \sqrt{3})(1 - \sqrt{3})} \\
= \frac{2\sqrt{3} - 4}{-2} = 2 - \sqrt{3}
\end{align*}
\]

In both pictures the obtuse angle is \( s + t, \) the short sides are radii of length 1, and the long sides have some common length \( d. \) Use the distance formula to calculate \( d. \)

In the first, 
\[
d = \sqrt{(\cos(s + t) - 1)^2 + (\sin(s + t) - 0)^2} \\
\therefore d^2 = \cos^2(s + t) - 2 \cos(s + t) + 1 + \sin^2(s + t) \\
\therefore d^2 = 2 - 2 \cos(s + t) \\
\text{since} \cos^2(s + t) + \sin^2(s + t) = 1.
\]

In the second, 
\[
d = \sqrt{(\cos t - \cos s)^2 + (\sin t + \sin s)^2} \\
\therefore d^2 = \cos^2 t - 2 \cos t \cos s + \cos^2 s \\
+ \sin^2 t + 2 \sin t \sin s + \sin^2 s \\
\therefore d^2 = 2 - 2 \cos t \cos s + 2 \sin t \sin s, \\
\text{since} \cos^2 t + \sin^2 t = 1, \cos^2 s + \sin^2 s = 1, \cos(-s) = \cos(s), \sin(-s) = -\sin(s).
\]

Equate the two terms which equal \( d^2: \)
\[
2 - 2 \cos(s + t) = 2 - 2 \cos t \cos s + 2 \sin t \sin s \\
\therefore -2 \cos(s + t) = -2 \cos t \cos s + 2 \sin t \sin s \\
\therefore \cos(s + t) = \cos \cos t - \sin s \sin t.
\]

**Proof of** \( \cos(s - t) = \cos s \cos t + \sin s \sin t. \)
\[
\cos(s - t) = \cos(s + (-t)) \\
= \cos s \cos(-t) - \sin s \sin(-t) \\
= \cos s \cos t + \sin s \sin t.
\]

**Proof of** \( \sin(s + t) = \sin s \cos t + \cos s \sin t. \)
\[
\sin(s + t) = \sin(\frac{\pi}{2} - (s + t)) = \cos((\frac{\pi}{2} - s) - t) \\
= \cos(\frac{\pi}{2} - s) \cos t + \sin(\frac{\pi}{2} - s) \sin t \\
= \sin s \cos t + \cos s \sin t.
\]

The other proofs are similar.
DOUBLE-ANGLE FORMULAS.

\[
\sin 2\theta = 2 \sin \theta \cos \theta \\
\cos 2\theta = \cos^2 \theta - \sin^2 \theta \\
\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}
\]

Proof.
\[
\sin 2\theta = \sin(\theta + \theta) \\
= \sin \theta \cos \theta + \cos \theta \sin \theta = 2 \sin \theta \cos \theta \\
\cos 2\theta = \cos(\theta + \theta) \\
= \cos \theta \cos \theta - \sin \theta \sin \theta = \cos^2 \theta - \sin^2 \theta \\
\tan 2\theta = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta} = \frac{2 \tan \theta}{1 - \tan^2 \theta}
\]

HALF-ANGLE FORMULAS.

\[
\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad \text{if } 0/2 \text{ is quadrant I or II, } - \text{if not.}
\]

\[
\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} \quad \text{if } 0/2 \text{ is quadrant I or IV, } - \text{if not.}
\]

\[
\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}
\]

Proof.
\[
\cos^2 \theta + \sin^2 \theta = 1. \text{ Hence}
\]

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta \\
= (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2 \sin^2 \theta
\]

\[
\therefore 2 \sin^2 \theta = 1 - \cos 2\theta
\]

\[
\therefore \sin \theta = \pm \sqrt{\frac{1 - \cos 2\theta}{2}}. \text{ Replacing } \theta \text{ by } \theta/2 \text{ gives}
\]

\[
\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}
\]

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta \\
= \cos^2 \theta - (1 - \cos^2 \theta) = 2 \cos^2 \theta - 1
\]

\[
\therefore 2 \cos^2 \theta = 1 + \cos 2\theta
\]

\[
\therefore \cos \theta = \pm \sqrt{\frac{1 + \cos 2\theta}{2}}. \text{ Replacing } \theta \text{ by } \theta/2 \text{ gives}
\]

\[
\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}
\]

\[
\tan \frac{\theta}{2} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \sqrt{\frac{1 - \cos \theta}{2}} \div \sqrt{\frac{1 + \cos \theta}{2}}
\]

\[
= \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right) \div \left( \frac{1 + \cos \theta}{1 + \cos \theta} \right)
\]

\[
= \sqrt{\frac{1 - \cos^2 \theta}{1 + \cos \theta}} = \sqrt{\frac{\sin^2 \theta}{1 + \cos \theta}} = \frac{\sin \theta}{1 + \cos \theta}
\]

Evaluate sin(2\theta), cos(2\theta), sin(\theta/2), cos(\theta/2).

First find sin(\theta). The inequality \(\theta < 0\Rightarrow \sin \theta > 0\).

\[
\sin \theta = \pm \sqrt{1 - \cos^2 \theta} = - \sqrt{1 - \frac{5}{9}} = - \frac{2}{3} \sqrt{2}
\]

\[
\sin 2\theta = 2 \sin \theta \cos \theta = 2 \left( -\frac{2}{3} \sqrt{2} \right) \left( \frac{1}{3} \right) = - \frac{4}{9} \sqrt{2}
\]

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \frac{1}{9} - \frac{8}{9} = - \frac{7}{9}
\]

\[
\frac{3\pi}{2} < \theta < 2\pi \Rightarrow \frac{3\pi}{4} < \frac{\theta}{2} < \pi
\]

\[
\Rightarrow \frac{\theta}{2} \text{ is II quadrant} \Rightarrow \sin \frac{\theta}{2} > 0, \cos \frac{\theta}{2} < 0
\]

\[
\sin \frac{\theta}{2} = \pm \sqrt{1 - \cos \frac{\theta}{2}} = \sqrt{1 - \frac{1}{3}} = \sqrt{\frac{2}{3}} = 1/\sqrt{3}
\]

\[
\cos \frac{\theta}{2} = \pm \sqrt{1 + \cos \frac{\theta}{2}} = - \sqrt{\frac{1}{3}} = - \sqrt{\frac{3}{3}} = - \sqrt{3}
\]

Evaluate using half-angle formulas.

\[
\sin \frac{\pi}{12} = \sin \frac{\pi/6}{2} = \sqrt{\frac{1 - \cos \frac{\pi}{6}}{2}} = \sqrt{\frac{1 - \sqrt{3}/2}{2}} = \frac{2 - \sqrt{3}}{4} = \frac{1}{2} \sqrt{2 - \sqrt{3}}
\]

\[
\cos \frac{\pi}{12} = \cos \frac{\pi/6}{2} = \sqrt{\frac{1 + \cos \frac{\pi}{6}}{2}} = \sqrt{\frac{1 + \sqrt{3}/2}{2}} = \frac{2 + \sqrt{3}}{4} = \frac{1}{2} \sqrt{2 + \sqrt{3}}
\]

\[
\tan \frac{\pi}{12} = \tan \frac{\pi/6}{2} = \frac{\sin \frac{\pi}{6}}{1 + \cos \frac{\pi}{6}} = \frac{1/2}{1} = \frac{1}{2 + \sqrt{3}}
\]

\[
= \frac{2 - \sqrt{3}}{(2 + \sqrt{3}) (2 - \sqrt{3})} = 2 - \sqrt{3}
\]

For the angle \(t\) in the picture, find \(\cos(2t)\) and \(\sin(t/2)\).

First find the missing side \(x\), then find \(\sin(t)\) and \(\cos(t)\).

\[
x^2 + 4 = 9, \quad x = \sqrt{9 - 4} = \sqrt{5}
\]

\[
\sin t = \sqrt{5}/3, \cos t = 2/3
\]

\[
2 \cos t = \cos^2 t - \sin^2 t = \frac{4}{9} - \frac{5}{9} = -\frac{1}{9}
\]

\[
\sin \frac{t}{2} = \pm \sqrt{\frac{1 - \cos t}{2}} = \sqrt{\frac{1 - 2/3}{2}} = \sqrt{\frac{3 - 2}{6}} = \frac{1}{\sqrt{6}}
\]

Express \(\sin^4 \theta\) in a form which does not use powers of trigonometric functions.

\[
\sin^4 \theta = (\sin^2 \theta)^2 = \left(1 - \cos 2\theta \right)^2 = 1 - 2 \cos 2\theta + \cos^2 2\theta
\]

\[
= 1 - 2 \cos 2\theta + \frac{1}{2} (1 + \cos 4\theta)
\]

\[
= \frac{1}{8} (3 - 4 \cos 2\theta + \cos 4\theta)
\]

The calculation for \(\cos^4 \theta\) is similar.
Product-to-sum rules

RECALL
\[
\cos(x - y) = \cos x \cos y + \sin x \sin y \\
\cos(x + y) = \cos x \cos y - \sin x \sin y
\]
Adding these gives
\[
\cos(x - y) + \cos(x + y) = 2 \cos x \cos y
\]
Subtracting gives
\[
\cos(x - y) - \cos(x + y) = 2 \sin x \sin y
\]
Similarly
\[
\sin(x - y) = \sin x \cos y - \cos x \sin y \\
\sin(x + y) = \sin x \cos y + \cos x \sin y
\]
Adding these gives
\[
\sin(x - y) + \sin(x + y) = 2 \sin x \cos y
\]
Dividing by 2 gives
\[
\text{PRODUCT-TO-SUM FORMULA.} \\
\sin x \sin y = \frac{1}{2} \left[ \cos(x - y) - \cos(x + y) \right] \\
\cos x \cos y = \frac{1}{2} \left[ \cos(x - y) + \cos(x + y) \right] \\
\sin x \cos y = \frac{1}{2} \left[ \sin(x + y) + \sin(x + y) \right]
\]
Write as a sum or difference of trigonometric functions.
\[
\begin{align*}
\sin \frac{\pi}{3} \cos \frac{\pi}{6} &= \frac{1}{2} \left[ \sin \left( \frac{\pi}{2} - \frac{\pi}{6} \right) \right] \\
&= \frac{1}{2} \left( \sin \frac{\pi}{3} + \sin \frac{\pi}{3} \right)
\end{align*}
\]
\[
\cos x \cos 2x = \frac{1}{2} [\cos(x - 2x) + \cos(x + 2x)] \\
= \frac{1}{2} [\cos(x) + \cos(3x)].
\]

Solving trigonometric equations

RECALL. sin \( x = \sin(\pi - x) \), cos \( x = \cos(-x) \).
Note: \( x \) and \( \pi - x \) are called supplementary angles.
Find all solutions for each equation.

\[
\begin{align*}
\sin \theta &= \frac{1}{2} \\
\theta &= \pi/6, 5\pi/6
\end{align*}
\]
The two simplest solutions: \( \theta = \pi/6, \) and \( \theta = -\pi/6 = 5\pi/6 \).
Since sin(\( \theta \)) has period 2\( \pi \), adding a multiple of 2\( \pi \) to either of these also produces a solution.
The general solution consists of the two sets
\[
\theta = \pi/6 + 2\pi n \text{ or } \theta = 5\pi/6 + 2\pi n \text{ for } n \text{ an integer.}
\]

\[
\begin{align*}
\cos x &= \frac{1}{2} \\
\text{Here we use } x \text{ for the angle instead of } \theta.
\end{align*}
\]
The two simplest solutions are \( x = \pi/3 \) and \( x = -\pi/3 \).
The general solution is
\[
x = \pi/3 + 2\pi n \text{ or } x = -\pi/3 + 2\pi n \text{ for } n \text{ an integer.}
\]

CONVENTION. Assume \( n \) is an arbitrary, possibly negative, integer. We’ll omit the phrase “for \( n \) an integer”.

\[
\begin{align*}
\tan x &= \frac{-1}{\sqrt{3}} \\
\text{Since } \tan \left( \frac{\pi}{6} \right) &= \frac{\sin \left( \frac{\pi}{6} \right)}{\cos \left( \frac{\pi}{6} \right)} = \left( \frac{1}{2} \right)\left( \frac{\sqrt{3}}{2} \right) = \frac{1}{\sqrt{3}}
\end{align*}
\]
\( \pi/6 \) is one solution. Since \( \tan(x) \) has period \( \pi \), adding a multiple of \( \pi \) also gives a solution. The general solution is \( x = \pi/6 + \pi n \).
For sin, cos, add \( 2\pi n \) to the (usually) two simplest solutions. For tan, add \( \pi n \) to the one simplest solution.

\[
\begin{align*}
\sin 2x &= 1 \\
\text{iff } 2x &= \pi/2 + 2\pi n, & \text{Only one simple solution here.} \\
\text{iff } x &= \pi/4 + \pi n.
\end{align*}
\]

RECALL. sin(\( x \)) and cos(\( x \)) are always between -1 and 1.
\[
\begin{align*}
\cos x &= 3 \\
\text{iff never. } & \text{no solution.}
\end{align*}
\]
\[
\begin{align*}
2 \cos^2 x + \cos x &= 1 \\
2 \cos^2 x + \cos x - 1 &= 0 \\
(2 \cos x - 1)(\cos x + 1) &= 0 \\
\text{cos } x &= \frac{1}{2} \text{ or } \cos x = -1 \\
x &= \pi/3 + 2\pi n \text{ or } x = -\pi/3 + 2\pi n \text{ or } x = \pi + 2\pi n.
\end{align*}
\]
\[
\begin{align*}
\cos^2 x - \cos x &= 2 \\
\cos^2 x - \cos x - 2 &= 0 \\
(\cos x + 1)(\cos x - 2) &= 0 \\
\cos x &= -1 \text{ or } \cos x = 2 \text{ The second is impossible.} \\
x &= \pi/2 + 2\pi n.
\end{align*}
\]
\[
\begin{align*}
2 \tan^2 x - 3 \tan x \sec x - 2 \sec^2 x &= 0 \\
2 \sin^2 x - 3 \sin x - 2 &= 0 \\
(2 \sin x + 1)(\sin x - 2) &= 0 \\
\sin x &= \frac{-1}{2} \text{ or } \sin x = 2 \text{ The second is impossible.} \\
x &= -\pi/6 + 2\pi n \text{ or } x = -5\pi/6 + 2\pi n.
\end{align*}
\]

RECALL. A function is 1-1 iff no horizontal line crosses its graph more than once.

sin, cos and tan are not 1-1. But they are 1-1 on the heavily marked intervals. These are the largest such intervals containing first quadrant angles.

For sin, the interval is \([ -\pi/2, \pi/2 ] \).
For cos, the interval is \([ 0, \pi ] \).
For tan, the interval is \([ -\pi/2, \pi/2 ] \).
Inverse trigonometric functions

**RECALL.** A function is 1-1 iff no horizontal line crosses its graph more than once.

While not 1-1 in general, sin, cos, and tan are 1-1 on the first quadrant angles in \([0, \pi/2] \) and on the larger heavily marked “1-1” intervals.

**DEFINITION.** \( \sin^{-1}, \cos^{-1}, \tan^{-1} \) are the inverses of the above restrictions of sin, cos, and tan. Thus 
\[
\sin^{-1}(x) = \theta \in [-\pi/2, \pi/2] \quad \text{such that} \quad \sin(\theta) = x,
\]
\[
\cos^{-1}(x) = \theta \in [0, \pi] \quad \text{such that} \quad \cos(\theta) = x,
\]
\[
\tan^{-1}(x) = \theta \in (-\pi/2, \pi/2) \quad \text{such that} \quad \tan(\theta) = x.
\]

\( \sin^{-1}(x) \) and \( \cos^{-1}(x) \) have domain \([-1, 1]\), for \( x \in [-1, 1] \), they are **undefined**. \( \tan^{-1}(x) \) has domain \(( -\infty, \infty)\).

**NOTATION.** \( \sin^{-1}(x), \cos^{-1}(x) \) and \( \tan^{-1}(x) \) are also written: \( \arcsin(x), \arccos(x), \) and \( \arctan(x) \).

**Warning.** \( \sin^{-1}(x) \neq 1/\sin(x) \). \( \sin^{-1}(x) \) is the inverse: \( (\sin(x))^{-1} = 1/\sin(x) \) is the reciprocal.

\[
\begin{align*}
\sin^{-1}(0) &= 0 & \cos^{-1}(0) &= \pi/2 & \tan^{-1}(0) &= 0 \\
\sin^{-1}(1/2) &= \pi/6 & \cos^{-1}(\sqrt{3}/2) &= \pi/6 & \tan^{-1}(1/\sqrt{3}) &= \pi/6 \\
\sin^{-1}(1/\sqrt{2}) &= \pi/4 & \cos^{-1}(1/\sqrt{2}) &= \pi/4 & \tan^{-1}(1) &= \pi/4 \\
\sin^{-1}(\sqrt{3}/2) &= \pi/3 & \cos^{-1}(\sqrt{3}/2) &= \pi/3 & \tan^{-1}(\sqrt{3}) &= \pi/3 \\
\sin^{-1}(1) &= \pi/2 & \cos^{-1}(0) &= \pi/2
\end{align*}
\]

The unit circle pictures are

\[
\begin{align*}
\text{The heavily marked half circles are the restricted ranges.}
\end{align*}
\]

**RECALL.** \( \cos(-x) = \cos(x), \quad \sin(\pi-x) = \sin(x) \).

When a formula involves an angle outside the restricted range, rewrite it in terms of an angle \( \theta \) in the restricted range before applying the equations in the box.

\[
\begin{align*}
\cos(\cos^{-1}(1/5)) &= 1/5 & \cos(\cos^{-1}(\pi/5)) &= \pi/5 \\
\cos(\cos^{-1}(-\pi/5)) &= \cos(\cos(\pi/5)) = \pi/5 & \cos(\cos(1/\sqrt{2})) &= \pi/5
\end{align*}
\]

| Note that | \( \sin(\sin^{-1}(1/2)) = -\pi/6 \) |
| \( \cos(\cos^{-1}(1/\sqrt{2})) = 3\pi/4 \) |
| \( \cos^{-1}(2) = \text{undef} \text{ since } 2 \notin [-1,1] \) |
| \( \arctan(-1) = -\pi/4 \) |
| \( \cos(\tan^{-1}(-1)) = \cos(-\pi/4) = \cos(\pi/4) = 1/\sqrt{2} \) |

Since inverses undo each other, we have

\[
\begin{align*}
\sin(\sin^{-1}(x)) &= x \text{ if } x \in [-1, 1] & \sin(\cos^{-1}(x)) &= x \text{ if } x \in [-1, 1] \\
\cos(\sin^{-1}(x)) &= x \text{ if } x \in [-1, 1] & \cos(\tan^{-1}(x)) &= x \text{ for any } x
\end{align*}
\]

**THEOREM.** For \( x \in [-1,1], \)
\[
\begin{align*}
\cos(\cos^{-1}(x)) &= \sqrt{1-x^2} & \sin(\sin^{-1}(x)) &= \sqrt{1-x^2}
\end{align*}
\]

Proof of 1st. Recall: \( \cos^2 \theta = 1 - \sin^2 \theta. \)

\[
\begin{align*}
\therefore \cos(\theta) &= \pm \sqrt{1 - \sin^2(\theta)} \\
\therefore \cos(\sin^{-1}(x)) &= \pm \sqrt{1 - (1 - (\sin(\sin^{-1}(x)))^2)}
\end{align*}
\]

The “+” was chosen over the “−” since \( \sin(\sin^{-1}(x)) = x \) if \( x \in [-1, 1] \) implies \( \cos(\sin^{-1}(x)) \geq 0. \)

**Example.**

\[
\begin{align*}
\sin(\sin^{-1}(1/2)) &= \frac{1}{2} \\
\cos(\cos^{-1}(1/\sqrt{2})) &= \sqrt{1 - \frac{1}{25}} = \sqrt{\frac{24}{25}} = \frac{2\sqrt{6}}{5} \\
\tan(\sin^{-1}(1/2)) &= \frac{\sin(\sin^{-1}(1/2))}{\cos(\sin^{-1}(1/2))} = \frac{1/2}{2\sqrt{6}/5} = \frac{\sqrt{6}}{12}
\end{align*}
\]

**Example.**

\[
\begin{align*}
\sec[\sin^{-1}(-1) - \cos^{-1}(1)] &= \sec[\pi/2 - 0] = 1/\cos(\pi/2) = 1/0 = \text{undef.}
\end{align*}
\]

In a triangle with hypotenuse 1 and side \( x, \)
\[
\begin{align*}
\sin^{-1}x &= \text{angle opposite } x, \\
\cos^{-1}x &= \text{angle adjacent to } x.
\end{align*}
\]

\[
\begin{align*}
\text{Note that } \sin^{-1}x + \cos^{-1}x = \pi/2 \quad (= 90^\circ)
\end{align*}
\]
Math 140  Lecture 25

Trigonometric word problems

(a) Find the exact answer.  (b) Find the decimal answer.

To test if your calculator is in radian or degree mode, calculate sin(1).
\[\sin(1^\circ) = .017, \sin(1\text{ rad}) = .84\]  Give only exact answers on tests.

The bookstore has cheap trig calculators for less than $20.

There are tables for sin and cos on pages A-34 to A-40.

Point P is level with the base of a 1000 ft building which is 4000 ft away. Find the angle of elevation from P to the top of the building, (a) the exact radian answer, (b) to nearest degree.

Let \(\theta\) be the angle.

\[\tan(\theta) = \frac{1000}{4000} = 1/4.\]

(a) \(\theta = \tan^{-1}(1/4)\)  \(\approx\)  to nearest degree.

(b)  \(\theta = 14^\circ\)

An antenna sits atop a 1000 ft building. From point a P on the ground, the angle of elevation to the top of the building is \(\beta\), the angle of elevation to the top of the antenna is \(\alpha\). Express the height of the antenna in terms of \(\alpha\) and \(\beta\).

Let \(x\) be the distance between P and the building. Then

\[\tan \beta = \frac{1000}{x}, \quad x = \frac{1000}{\tan \beta}\]

\[\tan \alpha = \frac{h+1000}{x}\]

\[x \tan \alpha = h + 1000\]

\[h = x \tan \alpha - 1000\]

Answer: Height is 1000(\(\tan \alpha / \tan \beta - 1\)) ft. Remember the units

From a point P level with the base of a mountain, the angle of elevation of the mountain is \(\alpha\). From a point Q 1 mile closer to the mountain’s base, the angle of elevation is \(\beta\). Express the height of the mountain in terms of \(\alpha\) and \(\beta\).

Let \(h\) be the height. Let \(x\) be as pictured.

Let \(h\) be the height. Let \(x\) be as pictured.

\[\tan \beta = \frac{h}{1+x}, \quad x = \frac{h}{\tan \beta}\]

\[\tan \alpha = \frac{h}{1+h/\tan \beta}\]

\[\frac{h}{1+h/\tan \beta} = \tan \alpha\]

\[h = (1 + \frac{h}{\tan \beta}) \tan \alpha\]

\[h \tan \beta = (\tan \beta + h) \tan \alpha\]

\[h \tan \beta = \tan \beta \tan \alpha + h \tan \alpha\]

\[h \tan \beta - h \tan \alpha = \tan \beta \tan \alpha\]

\[h(\tan \beta - \tan \alpha) = \tan \beta \tan \alpha\]

Answer height = \(\tan \beta \tan \alpha / (\tan \beta - \tan \alpha)\) miles

Theorem. The area of a triangle with sides \(a\) and \(b\) and included angle \(\theta\) is \(\frac{1}{2}ab \sin(\theta)\).

Proof. Case \(\theta\) is acute.

To find the area, we need to know the height \(h\).

\[\sin \theta = \frac{a}{h}\]

\[h = a \sin \theta\]

Area = \(\frac{1}{2}hb = \frac{1}{2}(a \sin \theta)b = \frac{1}{2}ab \sin \theta\)

Case \(\theta\) is obtuse.

Recall \(\sin(\pi - \theta) = \sin \theta\).

\[\sin(\pi - \theta) = \frac{h}{a}\]

\[h = a \sin(\pi - \theta) = a \sin \theta\]

Area = \(\frac{1}{2}hb = \frac{1}{2}(a \sin \theta)b = \frac{1}{2}ab \sin \theta\) (same answer)

Find the area of an octagon (stop sign) of radius 1 ft. Give the exact answer and the decimal answer to 2 places.

In each of the 8 triangles, the two sides are radii and have length 1.

\[\theta = \frac{2\pi}{8} = \frac{\pi}{4}\]

The area each triangle = \(\frac{1}{2}(1)(1)\sin(\frac{\pi}{4}) = \frac{1}{2}(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{4}\)

The area of the octagon = 8 x the area of each triangle

\[= 8 \frac{\sqrt{2}}{4} = 2\sqrt{2}\]

Exact answer = \(2\sqrt{2}\) sq.ft. Decimal answer = 2.83 sq. ft.
Today's problems involve 4 quantities; each is a side or an angle whose measure is given or wanted. Usually --

**Math 140   Lecture 26**

**CONVENTION.** Assume side \(a\) is opposite angle \(A\), side \(b\) is opposite angle \(B\) and side \(c\) is opposite angle \(C\).

**Sine laws.** In any triangle, the ratio of one angle's sine and its opposite side equals the ratio of any other angle's sine and opposite side.

Although written as one, there are 3 equations. Each involves two sides and two angles.

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

**Proof.** Recall that the area of a triangle is half the product of any two sides times the sine of the included angle. Thus the area of the triangle can be written three ways:

\[
\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C
\]

multiply by 2

\[
bc \sin A = ac \sin B = ab \sin C
\]

divide by abc

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

**Cosine laws.** For any two sides of a triangle, the sum of their squares minus twice their product times the cos of the included angle equals the square of the third side.

Each involves three sides and one angle.

\[
a^2 = b^2 + c^2 - 2bc \cos A
\]

\[
b^2 = a^2 + c^2 - 2ac \cos B
\]

\[
c^2 = a^2 + b^2 - 2ab \cos C
\]

**Proof.** We prove the last equality for the case \(C\) acute.

\[
\sin C = \frac{h}{a}, \quad \cos C = \frac{x}{a}, \quad \text{so} \quad h = a \sin C, \quad x = a \cos C
\]

\[
c^2 = (b-x)^2 + h^2
\]

\[
c^2 = b^2 - 2bx + x^2 + h^2
\]

\[
c^2 = b^2 - 2b(a \cos C) + a^2 \cos^2 C + a^2 \sin^2 C
\]

\[
c^2 = b^2 - 2ab \cos C + a^2 (\cos^2 C + \sin^2 C)
\]

\[
c^2 = a^2 + b^2 - 2ab \cos C
\]

**Straight angle sum fact.** The sum of a triangle’s 3 angles is a straight angle.

\[
\angle A + \angle B + \angle C = 180^\circ = \pi
\]

If you know two angles, you can solve for the third.

Solving for \(C\) gives \(\angle C = 180^\circ - (\angle A + \angle B)\).

Today’s problems involve 4 quantities; each is a side or an angle whose measure is given or wanted. Usually --

- For 2 sides and 2 angles: use the sine law involving the 2 sides (if necessary, get the third angle with the Straight Angle Sum Theorem).
- For 3 sides and 1 angle: use the cosine law involving the angle.

**Additional table-user step:**

\[
\sin x = \sin(\pi - x) = \sin(180^\circ - x), \quad \text{so} \quad \sin C = \sin 130^\circ = \sin(180^\circ - 130^\circ) = \sin 50^\circ.
\]

\[
\angle A = 20^\circ, \quad b = 50, \quad c = 60. \quad \text{Find} \ a.
\]

3 side, 1 angle problem with angle \(\angle A\).

Use the cosine law for \(\angle A\).

\[
a^2 = b^2 + c^2 - 2bc \cos A
\]

Solve for \(a\).

\[
a = \sqrt{b^2 + c^2 - 2bc \cos A}
\]

\[
= \sqrt{50^2 + 60^2 - 2(60)(50)\cos 20^\circ}
\]

\[
= \sqrt{6100 - 6000 \cos 20^\circ} \quad \leftarrow \text{Exact answer} = 21.49 \quad \leftarrow \text{2-place decimal answer}
\]

\[
a = 20, \quad b = 20, \quad c = 30. \quad \text{Find} \ C.A.
\]

3 side, 1 angle problem with angle \(\angle A\).

Use the cosine law for \(\angle A\).

\[
a^2 = b^2 + c^2 - 2bc \cos A
\]

Solve for \(\cos A\).

\[
2bc \cos A = b^2 + c^2 - a^2
\]

\[
\cos A = \frac{(b^2 + c^2 - a^2)}{2bc}
\]

\[
A = \cos^{-1}\left[\frac{(b^2 + c^2 - a^2)}{2bc}\right] = \cos^{-1}(900/1200) = \cos^{-1}(3/4) = 41.41^\circ
\]
A triangle is determined uniquely up to congruence (1) by two sides and an included angle and (2) by two angles and an included side.

However, given two sides and a noninclusion angle, there may be $0$, $1$, or $2$ triangles.

- Suppose $\angle A = 30^\circ$, $b = 2$. Then there is no triangle with $a = .5$, one triangle with $a = 1$, two triangles with $a = 1.5$, and one triangle with $a = 2.5$.

- Suppose two sides and a noninclusion angle are known. When solving for the third side using a cosine law, you may get an answer of the form $s \pm \sqrt{t}$. Then there is:
  - no solution if $t < 0$ or both $s \pm \sqrt{t} < 0$,
  - two solutions if $t > 0$ and both $s \pm \sqrt{t} > 0$,
  - one solution otherwise.

When solving for a second angle using a sine law, you may get an answer of the form $\theta = \sin^{-1}r, t \geq 0$. There is:
  - no solution if $t > 1$,
  - two solutions if $t < 1$ and the larger of the two sides is adjacent to the given angle (if one angle is $\theta$, the other $\pi - \theta$),
  - one solution otherwise.

- In a triangle, $\sin A = \frac{1}{2}$. What are the possible angles, in degrees, for $A$? One is $A = 30^\circ$, the other is $180^\circ - 30^\circ = 150^\circ$.
- Is there a triangle in which $a = 2, b = 3$, and $\angle A = 60^\circ$? Such a triangle exists iff the third side $c$ exists.

Solving for $c$.

\[
\begin{align*}
a^2 &= b^2 + c^2 - 2bc \cos A \\
4 &= 9 + c^2 - 2(3) c \cos 60^\circ \\
0 &= c^2 - 2(3) \frac{1}{2} c + 5 \\
c^2 - 3c + 5 &= 0 \\
c &= \frac{3 \pm \sqrt{3^2 - 4(1)(5)}}{2(1)} = \frac{3 \pm \sqrt{-11}}{2} = \text{undefined}
\end{align*}
\]

Since $c$ is undefined, the answer is no.

Try doing this and the next problem by drawing accurate pictures.

- $a = 2, b = 2, \angle A = 30^\circ$. Find $\angle C$ if $\angle B$ is acute.
  - First find $\angle B$ using the sine law, then find $\angle C$.
  - $\sin B = \sin A \cdot \frac{a}{b}, \sin B = \frac{b}{a} \sin A$. Thus one answer is $B = \sin^{-1} \left( \frac{a}{b} \sin A \right) = \sin^{-1} \left( \frac{2}{2} \sin 30^\circ \right) = \sin^{-1} \left( \frac{1}{2} \right) = 30^\circ$. The other answer is $180^\circ - 30^\circ = 150^\circ$. The acute angle is $30^\circ$.
  - $\angle C = 180^\circ - (\angle A + \angle B) = 180^\circ - (30^\circ + 30^\circ) = 120^\circ$.

### Polar coordinates

**Definition.** The polar coordinates $(r, \theta)$ of a point $P$ are its distance $r$ (radius) from the origin and the angle $\theta$ between the positive $x$-axis and the line from $(0,0)$ to $P$.

The usual $(x,y)$ are the rectangular coordinates.

Note, this is not the unit circle; it has radius $r$.

- Plot the points with polar coordinates $(1, \pi/4), (2, -\pi/2)$.
  - From the picture we have
    - $\sin \theta = \frac{y}{r}$
    - $\cos \theta = \frac{x}{r}$
    - $r^2 = x^2 + y^2$
    - $\tan \theta = \frac{y}{x}$
    - $y = r \sin \theta$
    - $x = r \cos \theta$
    - $r = \sqrt{x^2 + y^2}$
    - $\theta = \tan^{-1} \left( \frac{y}{x} \right)$
  - * Since $\tan$ has period $\pi$, $\theta$ and $\arctan(y/x)$ may differ by a multiple of $\pi$. Add $\pi$ if $(x,y)$ is in quadrant II or III. Here we suppose $r \geq 0$. 

**Theorem.** If a point has rectangular coordinates $(x, y)$ and polar coordinates $(r, \theta)$, then:

\[
(x, y) = (r \cos \theta, r \sin \theta) \quad \text{and} \quad (r, \theta) = \left( \sqrt{x^2 + y^2}, \arctan \left( \frac{y}{x} \right) \right) . \quad \text{Here } r \geq 0 .
\]

**Negative $r$.** $(-r, \theta)$ is the point $(r, \theta + \pi)$ on the opposite side of the origin $(0,0)$ as $(r, \theta)$.

- Convert from polar coordinates to rectangular: $(7, \frac{\pi}{6})$.
  - $r = 7, \theta = \pi/6$.
  - $(x, y) = (r \cos \theta, r \sin \theta) = (7 \cos \frac{\pi}{6}, 7 \sin \frac{\pi}{6}) = \left( \frac{7\sqrt{3}}{2}, \frac{7}{2} \right) \quad \leftarrow \text{answer}
  
- Convert from rectangular coordinates to polar: $(-1, \sqrt{3})$.
  - $x = -1, y = \sqrt{3}$. Point is in quadrant II, add $\pi$.
  - $(r, \theta) = \left( \sqrt{x^2 + y^2}, \tan^{-1} \left( \frac{y}{x} \right) \right) = \left( \sqrt{1 + 3}, \tan^{-1} \left( \frac{\sqrt{3}}{1} \right) \right) = (2, -\frac{\pi}{3}) \rightarrow * (2, \frac{2\pi}{3}) \quad \leftarrow \text{answer}
  
- Convert the polar equation to a rectangular equation: $r \sin \theta + 2 \cos \theta = 0$.
  - $y + 2(x/r) = 0 \rightarrow y + 2 \frac{x}{\sqrt{x^2 + y^2}} = 0$
  - $\sqrt{x^2 + y^2} y + 2x = 0 \quad \leftarrow \text{answer (hw simplifies more)}
  
- Convert the rectangular equation to a polar equation: $2x - y^2 = 0$.
  - $2r \cos \theta - r^2 \sin^2 \theta = 0 \quad \leftarrow \text{answer}
**Math 140  Lecture 28**

The graph of \( y = ax^2 + bx + c \) is a *vertical* parabola with a vertical axis of symmetry. Other parabolas have horizontal or slanted axes.

**Definition.** A parabola consists of all points equidistant between a given focus point and a given directrix line. The axis is the line through the focus and perpendicular to the directrix. The vertex is the intersection of the parabola and the axis.

In a parabola light rays parallel to the axis are reflected to the vertex. Telescope mirrors and satellite antennas have this shape. The vertex lies halfway between the focus and the directrix.

Find the equation for the parabola with focus (0, p) and directrix \( y = -p \).

For any point \((x, y)\),

The distance between \((x, y)\) and the directrix \(y = -p\) is \(y + p\).

The distance between \((x, y)\) and the focus \((0, p)\) is

\[
\sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + y^2 - 2py + p^2}.
\]

\((x, y)\) is on the parabola iff the distances are equal

- iff \( y + p = \sqrt{x^2 + y^2 - 2py + p^2} \)
- iff \( (y + p)^2 = x^2 + y^2 - 2py + p^2 \)
- iff \( y^2 + 2py + p^2 = x^2 + y^2 - 2py + p^2 \)
- iff \( 2py = x^2 - 2py \) iff \( 4py = x^2 \)
- iff \( x^2 = 4py \) iff \( x^2 = ky \) where \( k = 4p \) and \( p = k/4 \)

**Vertical Parabola Theorem.** For \( k \neq 0 \), the graph of \( x^2 = ky \) is the vertical parabola with focus \((0, p)\) and directrix \( y = -p \) where \( p = k/4 \). The axis is the \( y \)-axis; the vertex is \((0, 0)\).

Exchanging \( x \) and \( y \) gives —

**Horizontal Parabola Theorem.** For \( k \neq 0 \), the graph of \( y^2 = kx \) is a horizontal parabola with focus \((p, 0)\) and directrix \( x = -p \) where \( p = k/4 \). The axis is the \( x \)-axis; the vertex is \((0, 0)\).

**Theorem.** In any equation, replacing each

- \( x \) by \( x - a \) shifts the graph right by \( a \) units
- \( x \) by \( x + a \) shifts the graph left by \( a \) units
- \( y \) by \( y - b \) shifts the graph up \( b \) units
- \( y \) by \( y + b \) shifts the graph down \( b \) units

When a parabola is shifted, so are its focus, directrix, vertex, and axis.

To graph a parabola, get the squared variable on the left, the rest on the right. Complete the square if needed. Write the equations in the form:

\[(x \pm a)^2 = k(y \pm b) \text{ or } (y \pm b)^2 = k(x \pm a)\]

Find the focus, directrix and graph of \( x^2 - 2x + 9 - 8y = 0 \).

\[
x^2 - 2x = 8y - 9, \quad x^2 - 2x + 1 = 8y - 8, \quad \text{complete the square}
\]

\[(x - 1)^2 = 8(y - 1), \quad p = k/4 = 8/4 = 2. \text{ A vertical parabola.} \]

For \( x^2 = 8y \): vertex \((0, 0)\), focus \((0, 2)\), directrix \( y = -2 \).

To get \((x - 1)^2 = 8(y - 1)\), shift right 1 unit and up 1 unit.

Shifting \((0, 0)\) right 1 and up 1 gives the vertex: \((1, 1)\).

Shifting \((0, 2)\) gives the focus: \((1, 3)\).

Shifting \(y = -2\) gives the directrix: \( y = -1 \).
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- Find the focus, directrix and graph of \( y^2 + 2y = 4x - 5 \).
  \[ y^2 + 2y + 1 = 4x - 4, \quad (y + 1)^2 = 4(x - 1) \]
  \( 4p = 4, \quad p = 1 \).
  This is the parabola \( y^2 = 4x \) shifted: down 1 unit, right 1 unit.
  The \( y^2 \) means the parabola is horizontal.
  Shifting \((0,0)\) down 1 and right 1 gives the vertex: \((1, -1)\).

- The vertices are the two points farthest apart. The
  
  **THEOREM.** For \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is a horizontal ellipse: the foci are \((-c,0)\) and 
  \((c,0)\), the major axis is the line segment \((-a,0)(a,0)\), the
  minor axis is \((0,-b)(0,b)\).
  Here the bigger # is under \( x \).
  \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]
  is a vertical ellipse: the foci are \((0,-c)\) and 
  \((0,c)\), the major axis is the line segment \((0;-a)(0,a)\), the
  minor axis is \((-b,0)(b,0)\).
  Here the bigger # is under \( y \).
  
  To graph, complete the squares if necessary. Get 1 on the
  right. Write the equation in one of the forms above.

- Find the major and minor axes, the foci and draw the
  graph of \( 4x^2 + y^2 = 4 \).
  \[ x^2 + \frac{y^2}{4} = 1, \quad \frac{x^2}{1} + \frac{y^2}{2^2} = 1, \quad \frac{y^2}{2^2} + \frac{x^2}{1} = 1 \]
  \( a = 2, b = 1, c = \sqrt{4 - 1} = \sqrt{3} \)
  Vertical ellipse \( y \) has the bigger denominator.
  Major axis: \((0,2)(0,-2)\), minor axis: \((-1,0)(1,0)\),
  foci: \((0, -\sqrt{3})\), \((0, \sqrt{3})\).

- **RECALL.** A circle is the set of all points such that the
distance to a center point is some constant \( r \).

- **DEFINITION.** An **ellipse** is the set of all points such that the
sum of the distances to two **foci** points is some constant. The **major axis** goes from a vertex at one end of the ellipse through the two foci to the vertex at the opposite end. The **minor axis** is a perpendicular bisector of the major axis.

  The vertices are the two points farthest apart. The **major radius** \( a \) is half the major axis length; the **minor radius** \( b \) is half the minor axis length; the **focal radius** is half the distance between the foci.

  A light ray emitted from one focus point is reflected to the opposite focus point. Planetary orbits are ellipses.

- **THEOREM.** Let \( a, b, c \) be the major, minor and focal radii.
  Then \( a^2 = b^2 + c^2 \).

  The graph of \( x^2 + y^2 = r^2 \) is a circle with center \((0,0)\) and
  radius \( r \). This can be written as \( \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \).

- **THEOREM.** For \( a > b > 0 \), the graphs of \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) and
  \( \frac{y^2}{a^2} + \frac{x^2}{b^2} = 1 \) are ellipses. \((0,0)\) is the center. \( a, b, c \) are
  the major, minor and focal radii where \( c = \sqrt{a^2 - b^2} \).

- Shifting \((-3,0)\) right 3 gives the major axis: \((-2,0)(8,0)\).
  Shifting \((0,-4)(0,4)\) right 3 gives the minor axis: \((-3,4)(3,4)\).
  Shifting \((-3,0), (3,0)\) right 3 gives the foci: \((0,0), (6,0)\).

- Vertical ellipse \( x = -p \) gives the directrix: \( x = 0 \).
  Horizontal ellipse \( y = p \) gives the directrix: \( y = 0 \).
  
  Set your compass to a major radius. Put the
  point at the end of a minor radius. The two
  foci are where the arc of the compass
  intersects the major axis.
**DEFINITION.** A hyperbola is the set of all points such that the difference of the distances to two focus (or focal) points is some constant. The focal axis through the two foci intersects the hyperbola at its two vertices. The transverse axis is the segment between the vertices, the conjugate axis is a perpendicular bisector. The ends of the axes are the midpoints of a box whose diagonals are asymptotes. The box corners and the foci are equidistant from the center of the box. To get the foci, draw an arc around the box center from the box corner to the focal axis.

The path of a stone thrown upward (in a vertical gravitational field) is a parabola. The path of an orbiting planet or comet is an ellipse with the sun at one focus. The path of a comet passing through the solar system is one piece of a hyperbola with the sun at one focal point.

**THEOREM.** The graphs of \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) and \( \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \) are hyperbolas (a must be below the positive square).

- **a** = the transverse radius = \( \frac{1}{2} \) transverse axis length.
- **b** = the conjugate radius = \( \frac{1}{2} \) the conjugate axis length.
- **c** = \( \sqrt{a^2 + b^2} \)
  - the diagonal radius = \( \frac{1}{2} \) the length of a diagonal
  - the focal radius = \( \frac{1}{2} \) the distance between the foci.

- \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) is a horizontal hyperbola with
  - foci: \((-c,0)\) and \((c,0)\)
  - transverse axis: \((-a,0)(a,0)\)
  - conjugate axis: \((0,-b)(0,b)\)

- \( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \) is a vertical hyperbola with
  - foci: \((0,-c)\) and \((0,c)\)
  - transverse axis: \((0,-a)(0,a)\)
  - conjugate axis: \((-b,0)(b,0)\)

To graph, complete the squares if necessary, then write the equation in one of the above forms.

- Find the axes and foci and draw the graph and asymptotes of \( 4y^2 - x^2 = 4 \).
  
  \[
  \frac{y^2}{1} - \frac{x^2}{2} = 1
  \]
  
  \[
  a = 1, b = 2, c = \sqrt{1 + 4} = \sqrt{5}
  \]
  
  Vertical hyperbola (the \( y^2 \) is positive).
  - Transverse axis: \((0,-1)(0,1)\), conjugate axis: \((-2,0)(2,0)\), foci: \((0,\sqrt{5})\), \((0,-\sqrt{5})\).

- Find the axes and foci and draw the graph and asymptotes of \( 16x^2 - 96x - 25y^2 = 256 \).
  
  \[
  16(x^2 - 6x + \_ - 25y^2 = 256 \text{ complete square for } x}
  \]
  
  \[
  16(x^2 - 6x + 9) - 25y^2 = 256 + 16 \times 9
  \]
  
  \[
  16(x - 3)^2 - 25y^2 = 400 \text{ divide by } 400, \text{ get } 1 \text{ on right}
  \]
  
  \[
  \frac{(x - 3)^2}{25} - \frac{y^2}{16} = 1
  \]
  
  \[
  \frac{(x - 3)^2}{5^2} - \frac{y^2}{4^2} = 1
  \]
  
  \[
  a = 5, b = 4, c = \sqrt{25 + 16} = \sqrt{41}
  \]
  
  Horizontal hyperbola (the \( x^2 \) is positive).
  - This is \( \frac{x^2}{5^2} - \frac{y^2}{4^2} = 1 \) shifted right 3 units.

Shifting \((5,0)(5,0)\) right 3 gives the transverse axis: \((-2,0)(8,0)\).
Shifting \((0,-4)(0,4)\) right 3 gives the conjugate axis: \((3,4)(3,4)\).
Shifting \((0,-\sqrt{5}),(0,\sqrt{5})\) right 3 gives the foci: \((3,\sqrt{5}),(3,-\sqrt{5})\).