Math 311  Lecture 5

RECALL. A matrix is in reduced row echelon form (rref) iff
- the all-zero rows are at the bottom,
- the lead entry of a nonzero row is 1 and all other entries in its column are 0,
- no nonzero terms occur in the area below and left of a lead variable.

Examples of reduced row echelon matrices.

\[
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

RECALL. The elementary row (or just row) operations are:
- Interchange two rows.
- Multiply a row by a nonzero constant.
- Add to a given row, a multiple of another row.

Definition. Two matrices are row equivalent iff you can get from one to the other by a sequence of row operations.

Theorem. Using row operations, every matrix \( A \) can be reduced to a matrix \( \text{rref}(A) \) in reduced row echelon form.

Proof. Repeatedly applying row operations in the following order will convert any matrix to rref.
- Pick an unprocessed row with a leftmost nonzero entry.
- Make the entry 1.
- Move the row just above all other unprocessed rows.
- Make all other entries in the lead variable column 0.

Convert to rref with the above process.

\[
\begin{bmatrix}
0 & a & 2a-1 & 3a \\
5 & 7 & 9 & 11 \\
3 & 3 & 3 & 3
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & -3 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

Definition. A system of linear equations is homogeneous iff all right-hand constants are 0 iff its matrix form is \( AX = 0 \). Every homogeneous system has the trivial solution \( X = 0 \). A solution with one or more nonzero values is nontrivial.

\[\begin{align*}
x + y & -w = 0 \\
-2z & +4w = 0
\end{align*}\]

\[\begin{align*}
x = y = z = w = 0 \text{ is the trivial solution.}
\end{align*}\]

The rref form is:
\[\begin{align*}
x + y & -w = 0 \\
z & -2w = 0
\end{align*}\]

Writing the lead variables (\( x \) and \( z \) here) in terms of the remaining arbitrary variables gets the general solution.

The general solution is:

\[\begin{align*}
x = -y + w, & \quad z = 2w, \quad y, w \text{ arbitrary.}
\end{align*}\]

Picking a nonzero value for some arbitrary variable, say \( y = 1, w = 0 \), gives a nontrivial solution:
\[x = -1, \quad y = 1, \quad z = 0, \quad w = 0.
\]

The general solution can also be written parametrically:

\[x = -t + s, \quad z = 2s, \quad y = t, \quad w = s,
\]

where \( t \) and \( s \) are arbitrary parameters.

A system with \( k \) equations can have at most \( k \) lead variables, the rest will be arbitrary. Hence

Theorem. (a) A homogeneous system with more unknowns than equations has at least one arbitrary variable.
(b) If a system has an arbitrary variable, giving it a nonzero value gives a nontrivial solution.
(c) If a homogeneous system has no arbitrary variables, then it has no nontrivial solutions.

Find \( X = \begin{bmatrix} x \\ y \end{bmatrix} \) if \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and \( AX = X \)

\[AX = X \iff AX = IX \iff (A-I)X = 0 \iff \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Solving gives \( x = y \), \( y \) arbitrary. Hence

\[X = \begin{bmatrix} y \\ y \end{bmatrix}, \quad y \text{ arbitrary}, \quad \text{is the general solution.}
\]

How many solutions are there for \( 2x = 4 \)? \( 0x = 0 \)? \( 0x = 2 \)?

Theorem. \( ax = b \) has a unique solution if \( a \neq 0 \), infinitely many solutions if \( a = b = 0 \), no solution if \( a = 0 \) and \( b \neq 0 \).

\[\begin{align*}
x + 2y & -2z = 4 \\
-y & +5z = 2
\end{align*}\]

\[\begin{align*}
x + y & +(a^2 - 13)z = a + 2
\end{align*}\]

Find all \( a \) such that there is
(a) a unique solution,  (b) infinitely many solutions,  (c) no solution.

\[\begin{array}{ccc}
1 & 2 & -2 \\
0 & -1 & 5 \\
1 & 1 & a^2 - 13
\end{array}\]

\[\begin{array}{ccc}
x & y & z \\
1 & 0 & 8 \\
0 & 1 & -5 \\
0 & 0 & a^2 - 16
\end{array}\]

The last equation is \((a^2 - 16)z = a - 4\)
(a) unique: \(a \neq \pm 4\)  (b) many: \(a = 4\)  (c) none: \(a = -4\)