Math 311 Lecture 6

For today, all matrices will have \( n \) rows, for some fixed \( n \).
I means \( I_n \).

**Lemma.** If \( B \) is invertible and \( BA = I \), then \( A^{-1} = B \).

**Proof.** \( BA = I \) \( \implies B^{-1}BA = B^{-1} \) \( \implies A = B^{-1} \) \( \implies \), since being inverse is symmetrical, \( A^{-1} = B \).

Recall the three elementary row operations: swap two rows, multiply a row by a nonzero constant, and add a constant multiple of one row to another row.

**Definition.** If \( A \) and \( B \) have the same number of rows, \( A : B \) consists of the columns of \( A \) followed by the columns of \( B \).

**Lemma.** For any row operation \( e \), \( (eA : eB) = e(A : B) \).

**Lemma.** Every elementary row operation is invertible.

**Proof.** Swapping two rows twice, gives back the original matrix. The inverse of multiplying a row by \( a \) is multiplying it by \( a^{-1} \). The inverse of adding \( a \) times another row \( j \) is subtracting \( a \) times row \( j \).

**Definition.** For any row operation \( e \), let \( e(A) \) be the result of applying \( e \) to \( A \). \( e(A) \) is an **elementary matrix**.

**Lemma.** For any row operation \( e \), the elementary matrix \( e(A) \) is invertible. For any matrix \( A \), \( e(A) = e(A) \).

**Lemma.** A product of elementary matrices is invertible.

**Proof.** The product of invertible matrices is invertible.

- Let \( e = \) swap rows \( 1, 2 \). Then \( E = e(I) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).
- \( E(\begin{pmatrix} p & q \\ r & s \end{pmatrix}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} p & q \\ r & p \end{pmatrix} = e(\begin{pmatrix} p & q \\ r & p \end{pmatrix}) \).

Let \( f = \) multiply last row by \( a \). \( F = f(I) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \).

- \( F(\begin{pmatrix} p & q \\ r & s \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} p & aq + rs \\ ar + as & ps \end{pmatrix} = f(\begin{pmatrix} p & aq + rs \\ ar + as & ps \end{pmatrix}) \).

Let \( g = \) add \( a \) times first row to last row.

- \( G = g(I) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \).
- \( G(\begin{pmatrix} p & q \\ r & s \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} p & ap + rs \\ ar + aq + s \end{pmatrix} = g(\begin{pmatrix} p & ap + rs \\ ar + aq + s \end{pmatrix}) \).

Suppose applying a sequence \( e_1, e_2, e_3, \ldots, e_n \) of row operations to \( A \) gives \( B \). \( B = e_n(\ldots(e_3(e_2(e_1(A))))\ldots) = (e_n\ldots(e_2(e_1)))A \). Hence applying multiplying on left by \( (e_n\ldots(e_2(e_1)))A \) equals the result of applying the original sequence of operations.

**Theorem.** If a sequence of row operations converts \( (A : I) \) to \( (I : B) \), then \( B = A^{-1} \).

**Proof.** Suppose \( e_n(\ldots e_3(e_2(e_1(A : I))))\ldots) = (I : B) \).

- \( e_n(\ldots e_2(e_1)) = I \) & \( e_n(\ldots e_2(e_1(I))) = B \).
- \( (e_n\ldots e_2(e_1))A = I \) and \( (e_n\ldots e_2(e_1))I = B \). By the lemma above, \( A^{-1} = e_n\ldots e_2(e_1) = e_n\ldots e_2(e_1)I = B \).