**Math 311  Lecture 7**

Exam 1, Wednesday. Practice Exam Friday.

**LEMMA.** For any \( A \) and \( B \), if the \( i \)th row of \( A \) is all 0’s, then so is the \( i \)th row of \( AB \).

Hence a matrix with a row of 0’s is not invertible.

**PROOF.** Clear.

**LEMMA.** An \( n \times n \) \( rref \) matrix is either \( I_n \) (and hence is invertible) or has a row of 0’s (and hence is singular).

**PROOF.** Suppose the matrix is \( rref \). If the matrix is \( I_n \), all the leading entries are on the diagonal and the bottom 1 is in the rightmost column. Otherwise some leading entry is to the right of the diagonal. Hence so are all following leading entries and there can be no leading entry in the last row. Thus the last row is all 0.

**THEOREM.** For any \( n \times n \) matrix \( A \), the following are equivalent:

1. \( A \) is invertible.
2. \( AX = 0 \) has only the trivial solution.
3. \( AX = B \) has a unique solution for any column matrix \( B \).
4. \( A \) is row equivalent to \( I_n \).
5. \( A \) is a product of elementary matrices.

**PROOF.**

1 \( \Rightarrow \) 2. If \( A^{-1} \) is the inverse of \( A \), then \( AX = 0 \) implies \( A^{-1}AX = A^{-1}0 \) implies \( X = 0 \).

2 \( \iff \) 3 \( \iff \) 4. System has a unique solution iff the coefficient part, i.e., \( A \), reduces to the identity matrix (if the last coefficient is not 1, it is 0 and the last equation is \( 0 = \text{nonzero or } 0 = 0 \) and there are no solutions or infinitely many solutions).

4 \( \Rightarrow \) 5. Proved last time.

5 \( \Rightarrow \) 1. Proved last time. \( \square \)

**COROLLARY.** For any \( n \times n \) matrix \( A \), the following are equivalent:

1. \( A \) is singular.
2. \( AX = 0 \) has a nontrivial solution.
3. \( A \) is row equivalent to a matrix with a row of zeros.

**LEMMA.** If \( A \) is singular, so is \( AB \) for any \( B \).

**PROOF.** If \( A \) is singular, it is row reducible to a matrix \( EA \) with a row of 0’s where \( E \) is the product of elementary matrices which does the reduction. Thus the product \( EAB \) also has a row of 0’s. Thus \( AB \) is row equivalent to a matrix with rows of zeros. Thus \( AB \) is singular.

**THEOREM.** For \( n \times n \) matrices, if \( AB = I_n \), then \( B = A^{-1} \).

**PROOF.** If \( A \) were singular, then \( AB = I_n \) would also be singular which it isn’t. Hence \( A \) is invertible and, by last time’s lemma, \( B = A^{-1} \).

\( \square \)

**DEFINITION.** Elementary column operations are defined the same way as row operations. You can swap two columns, multiply a column by a nonzero constant, add to one column a constant multiple of another.

**DEFINITION.** For any column operation \( f \), let \( (A)f \) be the result of applying the operation to \( A \). \( F = (I_n)f \) is the elementary matrix for \( f \).

**THEOREM.** For any column operation \( f \) with elementary matrix \( F \), for any matrix \( A \), \( (A)f = AF \).

\( f = \text{add } a \text{ times last column to first}, \ A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \). Then

\[
F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad f = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}
\]

\[
AF = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p + aq & q \\ r + as & s \end{bmatrix} = (A)f
\]

**THEOREM.** With row and column operations, every matrix can be reduced to a matrix consisting of an identity matrix plus possible additional all 0 zero rows at the bottom and/or 0-columns at the right.

**PROOF.** Reduce the matrix to row echelon form. Get rid of any nonzero nonleading entries with column operations. Swap columns to get the leading entries on the diagonal. Reductions don’t have to be in the above order. \( \square \)

**Reduce the following.**

\[
\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

**HW 5 Answers**

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2b(2). \( I_n \)

6a(2). \( x = -2 + w, \quad y = -1, \quad z = 8 - 2w, \quad w \) arbitrary.

8a(2).

\[
\begin{array}{c|c|c|c|c}
\text{z} & \text{y} & \text{z} & \text{w} \\
\hline
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{array}
\]

\( x = 1 - w, \quad y = 2, \quad z = 1, \quad w \) arbitrary.

8b(2).

\[
\begin{array}{c|c|c|c|c}
\text{z} & \text{y} & \text{z} & \text{w} \\
\hline
1 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & -1 \\
\end{array}
\]

\( x = 1 - w, \quad y = 2 + w, \quad z = -1 + w, \quad w \) arbitrary.

10(2). \( x = \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad 12(2). \ X = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ z \end{bmatrix} \)

14(2). (a) \( a = -2, \) (b) \( a \neq 2, \) (c) \( a = 2 \)

16(2). (a) \( a = \pm \sqrt{6}, \) (b) \( a \neq \pm \sqrt{6}, \) (c) none

24(1). \( I_2 \)