Math 311  Lecture 10

Subspaces
A subset W of a vector space V is also a vector space under the operations of V iff 0∈W and W is closed under addition and scalar multiplication, i.e., if u,v∈W and c∈R implies u+v∈W and cu∈W.

Definition. W is a subspace of a vector space V iff W⊆V, and for all u,v∈W and all scalars c:
0∈W, u+v∈W, and cu∈W.
The last two conditions alone suffice since cu∈W for all c ⇒ 0∈W ⇒ 0∈W.

For any vector space V, {0} is a subspace, called the zero subspace. Since V⊆V, V is also a subspace of itself.

Write “subspace” if the given set W is a subspace.

Either, otherwise
(1) note that 0∈W or
(2) find u,v∈W such that u+v∉W or
(3) find u∈W and a scalar c such that cu∉W.

W={a
2a : a,b∈R}

W={a
b : a>0}

W={a,b,c,d: a>0, b<0}

Null Spaces
Definition. For any matrix A, the null space of A is the set of vectors X such that AX=0.

Lemma. The null space of a matrix A is a subspace.

Proof. A0=0 ⇒ 0∈null space.

Suppose u, v∈null space, want u+v, cu∈null space.
A(u+v)=Au+Av = 0+0 = 0 ⇒ u+v∈the null space.
A(cu)=c(Au) = c0 = 0 ⇒ cu∈null space.

Find the null space for A=[1 1 2 2].

A[x
y]=[0 0] iff x+y = 0 and 2x+2y = 0 iff x+y = 0
iff x = -y. Hence the null space of A is \{[-y
y]: y∈R\}.

Linear combinations and spans
Recall. v is a linear combination of v₁, v₂, v₃, ... iff
v = av₁+bv₂+cv₃+... for some scalars a, b, c, ... .

Definition. The span a set S = {v₁, v₂, v₃, ... } of vectors, is the set of all linear combinations of v₁, v₂, v₃, ...

Theorem. The span of a set of vectors is a subspace.

Proof. As noted above, it suffices to check just the conditions: u, v∈W and c a scalar ⇒ u+v∈W, cu∈W.
A sum of linear combinations is also a linear combination.
A scalar multiple of a linear combination is also a linear combination.

Given v₁=[1, 1, 0], v₂=[0, 1, 1], v₃=[1, 0, -1], write the following as linear combinations of v₁, v₂, v₃ if possible, otherwise write ∗.
(a) v=[1, 3, 2]
(b) v=[1, 1, 1].

Solution. It is easier to solve the equivalent column vector version.
(a) v is a linear combination of v₁, v₂, v₃ iff xv₁+yv₂+zv₃ = v for some scalars x, y, z.
iff x, y, z is a solution of
x 1+y 1+z 0 = 1, x +y 3 = 3, y −z 2 = 2
Reducing the augmented matrix gives
\[
\begin{bmatrix}
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 3 \\
0 & 1 & -1 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & -1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
∴ x = 1−z, y = 2+z, z arbitrary.

To get a particular combination, set the arbitrary z to 0. This gives x=1, y=2, and z=0.
∴ [1, 3, 2]=v=xv₁+yv₂+zv₃=1v₁+2v₂+0v₃=v₁+2v₂.

(b) For v=[1, 1, 1] the matrices are:
\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
which has no solutions.

Thus v = [1, 1, 1] is not a linear combination. ∗

Parametric lines
Lemma. In 3-dimensional space R³, (a similar result holds for other dimensions) for any point p = (x₀, y₀, z₀).

and any vector v = [a
b ] , the line through p which is parallel to v is the set of points (x, y, z) with the following parametric equation:

x = x₀ + at, y = y₀ + bt, z = z₀ + ct, t arbitrary.

Find the parametric equation for the line through
(1,2,3) which is parallel to [6
5 ] .

x = 1+6t, y = 2+5t, z = 3+4t.