Math 311   Lecture 12

DEFINITION. Vectors \( v_1, v_2, \ldots, v_n \) are a basis for a vector space \( V \) iff they span \( V \) and they are independent.

- The standard basis for \( \mathbb{R}^3 \) is \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). Clearly the three vectors span \( \mathbb{R}^3 \) and are independent.
- The standard basis for \( \mathbb{P}_2 \) is \( \{ t^2, t, 1 \} \).

The standard basis for other spaces is defined similarly.

THEOREM. If \( v_1, v_2, \ldots, v_n \) are a basis for \( V \), then every vector of \( V \) can be written in one and only one way as a linear combination of \( v_1, v_2, \ldots, v_n \).

PROOF. Suppose \( v \) can be written as two linear combinations
\[ a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \quad \text{and} \quad b_1 v_1 + b_2 v_2 + \cdots + b_n v_n = \]
\[ (a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \cdots + (a_n - b_n) v_n = 0. \]
By linear independence, \( (a_1 - b_1) = (a_2 - b_2) = \cdots = (a_n - b_n) = 0. \)
Hence the linear combinations must be the same.

RECALL. If a set of vectors is dependent, then one can be written as a linear combination of the others.

THEOREM. If \( v_1, v_2, \ldots, v_n \) span \( W \), then some subset of \( v_1, v_2, \ldots, v_n \) is a basis for \( W \).

PROOF. If \( v_1, v_2, \ldots, v_n \) are not independent, then one can be written in terms of the others. Delete this vector. If the result is still dependent, then again one can be written in terms of the others, and again delete it. Repeat the process until the set becomes independent.

Here is the systematic way to do this.

**LEMMA.** Reduce the matrix \( A = (v_1 \mid v_2 \mid \ldots \mid v_n) \) to ref. Then the vectors \( v_i \) whose columns in the ref matrix have leading 1’s form a basis for the span \( W \) of \( v_1, v_2, \ldots, v_n \).

**PROOF.** Suppose \( n = 4 \). Consider the homogeneous system:
\[
\begin{align*}
x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 &= 0. \\
\end{align*}
\]
Suppose that in the rref matrix, just the columns for \( v_1 \) and \( v_2 \) have leading 1’s. Then in the general solution, \( x_3 \) and \( x_4 \) are arbitrary variables. We claim that \( v_2 \) and \( v_3 \) are dependent.

To write \( v_2 \) in terms of \( v_1 \) and \( v_4 \), set \( x_3 = 1 \) and \( x_4 = 0. \) Thus \( x_1 v_1 + 1 v_2 + 0 v_3 + x_4 v_4 = 0. \) Thus \( v_2 = -x_1 v_1 - x_4 v_4. \)

To write \( v_2 \) in terms of \( v_1 \) and \( v_4 \), set \( x_2 = 1 \) and \( x_3 = 0. \)

Delete the dependent vectors \( v_2 \) and \( v_3 \) to get the basis \( \{ v_1, v_4 \} \).

With this procedure the deleted vectors will, in fact, depend only on earlier undeleted vectors. Hence \( v_2 \) and \( v_3 \) will be written in terms of just the earlier vector \( v_1 \).

Find a basis for the subspace \( W \) spanned by the vectors
\[ v_1 = [1,-1,0], \quad v_2 = [-1,1,0], \quad v_3 = [1,0,1], \quad v_4 = [0,1,1]. \]

First convert to the equivalent column vector problem. Second, eliminate the dependent vectors.

To find the dependencies, solve the homogeneous system of equations:
\[
\begin{pmatrix}
x_1 + y v_2 + z v_3 + w v_4 = 0. \\
\end{pmatrix}
\]
Reducing the augmented matrix to rref gives:
\[
\begin{pmatrix}
1 & -1 & 1 & 0 \\
-1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -1 & 0 & -1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
and hence we get
\[ x = y + w, \quad z = -w, \quad y, w \text{ arbitrary.} \]

By the lemma, the original vectors whose rref columns have leading 1’s are a basis. Converting back to rows gives:
Answer: \( v_1 = [1,-1,0] \) and \( v_3 = [1,0,1] \) are a basis.

To write \( v_4 \) in terms of \( v_1 \) and \( v_3 \), set the variable \( y \) for \( v_2 \) to 1, set \( w = 0. \)

Then \( x = y + w = 1 + 0 = 1, \) \( z = -w = 0. \)

Hence \( x v_1 + y v_2 + z v_3 + w v_4 = 0 \) becomes
\[ 1 v_1 + 1 v_2 + 0 v_3 + 0 v_4 = 0. \]
Solve for \( v_2: v_2 = -v_1. \)

Likewise \( v_4 \) can also be written as a linear combination of \( v_1 \) and \( v_3. \) Set \( w = 1, y = 0. \) \( x = y + w = 0 + 1 = 1, \) \( z = -w = -1. \)

Hence \( 1 v_1 + 0 v_2 + (-1) v_3 + 1 v_4 = 0. \) So \( v_4 = v_3 - v_1. \)

Find a basis for the subspace of all skew symmetric matrices of \( \mathbb{R}_{3x3}. \)

An arbitrary such matrix looks like
\[ \begin{pmatrix}
0 & a & b \\
-a & 0 & c \\
-b & -c & 0 \\
\end{pmatrix}. \]

Separate the parts with \( a, b, c, \) then factor out \( a, b, c: \)
\[ \begin{pmatrix}
0 & a & b \\
-a & 0 & c \\
-b & -c & 0 \\
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 \\
-a & 0 & 0 \\
-b & -c & 0 \\
\end{pmatrix} + \begin{pmatrix}
a & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} + \begin{pmatrix}
b & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}. \]

Hence
\[ \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \quad \text{is a basis.} \]

In \( \mathbb{P}_3, \) find a basis for the subspace of polynomials of the form \( (a+b)t^2 + (a+b)t + b + c. \)

Separating the parts gives that any such polynomial can be written as the sum
\[
(a+b)t^2 + (a+b)t + c = (at^2 + a) + (bt^2 + b + a) + (c)
\]
\[ = a(t^2 + t) + b(t^2 + t + 1) + c(1). \]

Hence any such polynomial can be written as a sum of the three polynomials \( t^2 + t, \quad t^2 + t + 1, \quad 1. \)

Are these independent? No, \( t^2 + t + 1 \) is the sum of \( t^2 + t \) and \( 1. \)

Deleting this dependent vector gives \( \{ t^2 + t, 1 \} \) which is independent and hence a basis.

Find a basis for the subspace \( V = \{ [a, b, c] : \quad a = b + c \}. \)

Replacing \( a \) by \( b + c \) gives \( [a, b, c] = [b + c, b, 0] + [c, 0, 0] = b[1, 1, 0] + c[1, 0, 1]. \)

Hence every vector in \( V \) is a linear combination of \( [1, 1, 0] \) and \( [1, 0, 1] \).