**Math 311    Lecture 17**

**Assumption for the Day.** Assume all vectors are either row vectors in \( \mathbb{R}^n \) or column vectors in \( \mathbb{R}^n \) for some \( n \).

**Definition.** For any row vectors \( u = [a_1, a_2, \ldots, a_n] \), \( v = [b_1, b_2, \ldots, b_n] \), \( u \cdot v \), the inner product (also called the standard inner product or dot product) of \( u \) and \( v \) is the scalar \( a_1b_1 + a_2b_2 + \ldots + a_nb_n \). The inner product of column vectors is defined the same way.

- If \( u = [1,0,0], \ v = [0,1,0] \), then \( u \cdot v = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0 \).

If \( u = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \ v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \), then \( u \cdot v = 2 \cdot 2 + 2 \cdot 3 = 12 \).

Suppose \( u = [a,b]^T \) is a vector of \( \mathbb{R}^2 \). Then the length of \( u \) is:

\[
\sqrt{a^2 + b^2}
\]

Note that \( u \cdot u = a \cdot a + b \cdot b \). Hence the length is \( \sqrt{u \cdot u} \).

**Definition.** For any row or column vector \( u \), the length of \( u \) is \( ||u|| = \sqrt{u \cdot u} \).

Note in the picture above, the distance between the heads of \( u \) and \( v \) is the length of \( u - v \). Hence, in general:

**Definition.** The distance between \( u \) and \( v \) is \( ||u - v|| \).

- Find the distance between \( u = [1,2,3,4] \) and \( v = [4,3,2,1] \).
  
  Distance \( ||u - v|| = ||[-3,-1,1,3]|| = \sqrt{(-3)^2 + (-1)^2 + 1^2 + 3^2} = \sqrt{20} = 2\sqrt{5} \)

Returning to the picture above, by the law of cosines we have:

\[
||u - v||^2 = ||u||^2 + ||v||^2 - 2||u|| \cdot ||v|| \cos \theta
\]

\[
\therefore (u - v) \cdot (u - v) = u \cdot u + v \cdot v - 2||u|| \cdot ||v|| \cos \theta
\]

\[
\therefore u^2 - 2u \cdot v + v^2 = u^2 + v^2 - 2||u|| \cdot ||v|| \cos \theta
\]

\[
\therefore -2u \cdot v = -2||u|| \cdot ||v|| \cos \theta
\]

\[
\therefore u \cdot v = ||u|| \cdot ||v|| \cos \theta
\]

\[
\therefore \cos \theta = \frac{u \cdot v}{||u|| \cdot ||v||}
\]

**Theorem.** The cosine of the angle between \( u \) and \( v \) is \( \frac{u \cdot v}{||u|| \cdot ||v||} \).

Two nonzero vectors \( u \) and \( v \) are perpendicular or orthogonal iff the angle between them is \( \pm \pi/2 \).

iff \( \cos \theta = 0 \) iff \( \frac{u \cdot v}{||u|| \cdot ||v||} = 0 \) iff \( u \cdot v = 0 \).

**Definition.** Vectors \( u \) and \( v \) are orthogonal iff \( u \cdot v = 0 \).

- Find \( a \) such that \( u = [a,2] \) is orthogonal to \( v = [-1,2] \).
  
  \( u \) is orthogonal to \( v \) iff \( u \cdot v = 0 \) if \( a(-1) + 2 \cdot 2 = 0 \) iff \( -a + 4 = 0 \) if \( a = 4 \). Answer: \( a = 4 \).

**Definition.** To vectors are in the same direction if one is a positive multiple of the other, they are in opposite directions if one is a negative multiple of the other, they are parallel if one is a nonzero multiple of the other iff they are in the same or opposite directions.

- Which pairs are in the same direction? are in opposite directions? are parallel? are orthogonal?
  
  \( x = [1,0,1], \ y = [2,0,2], \ z = [1,3,-1], \ w = [-2,-6,2] \).

  Same direction: \( (x, y) \)
  
  Opposite direction: \( (z, w) \)
  
  Parallel: \( (x, y), (z, w) \)
  
  Orthogonal: \( (x, z), (x, w), (y, z), (y, w) \).

**Definition.** \( u \) is a unit vector iff it has length 1 iff \( ||u|| = 1 \).

For any \( u, u/||u|| \) is a unit vector in the same direction as \( u \).

- Find the unit vector in the direction of \( u = [1,1] \).
  
  \( u/||u|| = \frac{[1,1]}{\sqrt{1^2 + 1^2}} = \frac{[1,1]}{\sqrt{2}} = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \).

- Find \( c \) if \( c[-2] \) is in the same direction as \( [-5,-10] \).
  
  If they are in the same direction, their components are proportional, e.g., \( [a, b] = [ka, kb] \). Thus corresponding components have the same ratio, e.g., \( \frac{a}{ka} = \frac{b}{kb} = \frac{1}{k} \).

Hence \( c = \frac{-2}{-10} \cdot \ldots \cdot c = -1 \).

**Resultants**

Velocities and forces are vectors. The result of combining two velocities or two forces is usually the sum of the two vectors. If the velocity is \( v \), then \( ||v|| \) is the speed. If \( v \) is a force, \( ||v|| \) is the magnitude of the force.

- A plane heads flies at an airspeed of 500 mph with its nose pointed north. If there is a 20 mph wind blowing to the west, what is the plane’s resultant speed?

Identifying north with the y-axis and east with the x-axis, the plane’s velocity through the air is \( [0, 500] \), the velocity of the wind is \( [-20, 0] \). Hence the resultant velocity is \( [0, 500] + [-20, 0] = [-20, 500] \). The resultant speed is \( ||[-20, 500]|| = \sqrt{(-20)^2 + 500^2} = 20\sqrt{626} \).

**On homework and tests, give exact answers such as the radical above, not decimal answers such as 500.3998.**