Assumption for the day. All vectors are in $\mathbb{R}^3$.

Let $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence a vector $v \in \mathbb{R}^3$ can be written variously as

$$v = [v_1, v_2, v_3]^T = v_i + v_j + v_k.$$

**Definition.** For $v = [a, b, c]^T$ and $v' = [a', b', c']^T$, let $v \times v'$, the cross product of $v$ and $v'$, equals $[bc' - cb', ca' - ac', ab' - ba']^T$.

If $v = [1, 2, 3]^T$ and $u = [4, 5, 6]^T$, then $v \times u = [2 - 6 - 35, 3 - 4 - 16, -1 - 5 - 24] = [-36, -3]$. 

**Lemma.** For any $u, v \in \mathbb{R}^3$, $u \times v$ is orthogonal to $u$ and to $v$.

**Proof.** If $u = [a, b, c]^T$ and $v = [a', b', c']^T$ then $u \times v = [a, b, c]^T \times [bc' - cb', ca' - ac', ab' - ba']^T = abc' - abc' + bca' - acb' + cab' - cba' = 0$. 

Note. $u \times u = 0$. $u \times u = [bc - cb, ca - ac, ab - ba]^T = [0, 0, 0]^T$. 

$i \times j = [1, 0, 0]^T \times [0, 1, 0]^T = [0, 0, 1]^T = k$. Likewise $j \times k = i$ and $k \times i = j$. However $j \times i = -k$, $k \times j = -i$, $i \times k = -j$.

If $i, j, k$ are put in a circle, then the cross product of clockwise successive vectors is the next vector; of counterclockwise successive vectors is the negative of the next vector.

Cross product has the distributivity properties but associativity fails and it is **anticommutative**.

**Basic properties of cross product.**

$u \times v = -v \times u$

$u \times (v+w) = u \times v + u \times w$

$(u+v) \times w = u \times w + v \times w$

$c(u \times v) = (cu) \times v = u \times (cv)$

$u \times u = 0$

$0 \times u = u \times 0 = 0$

$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$

$(u \times v) \times w = (v \cdot u)v - (v \cdot w)u$

$(u \times v) \cdot w = u \cdot (v \times w)$

$(u \times v) \cdot w$ and $(u \times v) \cdot w$ are called the **triple product**.

$u \times v$ is perpendicular or **normal** to the plane of $u$ and $v$. It points in the direction a right-hand screw would go along the line perpendicular to the plane of $u$ and $v$ if it were rotated through the angle $\theta$ from $u$ to $v$. This explains the $u \times v = -v \times u$, $u \times (v \times w)$, and $(u \times v) \times w$ rules.

**Recall.** $u \cdot v = ||u|| \cdot ||v|| \cos \theta$ since $\cos \theta = u \cdot v / (||u|| \cdot ||v||)$.

**Lemma.**$|u \times v| = ||u|| \cdot ||v|| \sin \theta$ is the area of the parallelogram with sides $u$ and $v$.

**Proof.** The property $(u \times v) \cdot w = u \cdot (v \times w)$, with $w = [u \times v]$, gives $(u \times v) \cdot [u \times v] = u \cdot (v \times [u \times v])$. Thus $||u \times v||^2 = (u \times v) \cdot (u \times v) = u \cdot (v \times (u \times v)) = u \cdot (v \cdot (u \times v)) - (v \cdot (u \times v)) (u \cdot (u \times v))$

$|u \times v|^2 - (u \times v) \cdot (v \times [u \times v]) = ||u||^2 ||v||^2 - (||u|| \cdot ||v|| \cdot \cos \theta)^2 = ||u||^2 ||v||^2 - ||u \times v||^2 \cos \theta = ||u||^2 ||v||^2 [1 - \cos^2 \theta] = ||u||^2 ||v|| \sin \theta$. Taking square roots gives the answer.

In the picture, $\sin \theta = h / ||v||$; hence the height $h = ||v|| \sin \theta$.

The area of the parallelogram (twice the area of the triangle) $= ||u|| \cdot h = ||u|| \cdot ||v|| \sin \theta = ||u \times v||$.

**Lemma.** The volume of the parallelepiped formed by three vectors $u, v, w$ is the triple product $[u \cdot (v \times w)]$.

- Find the area of the triangle with vertices $(1, 1, 1)$, $(1, 1, 6)$, $(1, 2, 2)$.

  Let $u$ be the vector from $(1, 1, 1)$ to $(1, 1, 6)$ $=[0, 0, 5]^T$.

  Let $v$ be the vector from $(1, 1, 1)$ to $(1, 2, 2)$ $=[0, 1, 1]^T$.

  Then $u$ and $v$ are two sides of the triangle and the area $= \frac{1}{2} ||u \times v|| = \frac{1}{2} ||[0, 0, 5]^T \times [0, 1, 1]^T|| = \frac{1}{2} ||[-5, 0, 0]^T|| = 5 / 2$.

- Find an equation for the plane which passes through the three points $(1, 1, 1), (1, 1, 6)$ and $(1, 2, 2)$.

  Let $u$ and $v$ be the vectors from $(1, 1, 1)$ to the other points as above. Then $u$ and $v$ lie in the plane and $u \times v = [-5, 0, 0]^T$ is perpendicular to the plane. Hence $(x, y, z)$ is in the plane if and only if the vector $[x - 1, y - 1, z - 1]^T$ from $(1, 1, 1)$ to $(x, y, z)$ lies in the plane of $u$ and $v$ if and only if it is perpendicular to $u \times v$ if $[x - 1, y - 1, z - 1]^T \cdot (u \times v) = 0$ if $x - 1 = 0$ if $y - 1 = 0$ if $z - 1 = 0$.

  The equation for the plane is: $x = 1$.

- Find an equation for the plane which passes through $(1, 2, 3)$ and is perpendicular to $[5, 4, 3]^T$.

  $(x, y, z)$ is on this plane if and only if the vector from $(1, 2, 3)$ to $(x, y, z)$ is perpendicular to $[5, 4, 3]^T$ if and only if $[x - 1, y - 2, z - 3]^T \cdot [5, 4, 3]^T = 0$ if $5(x - 1) + 4(y - 2) + 3(z - 3) = 0$ if $5x + 4y + 3z = 22$.

  Answer: The equation is $5x + 4y + 3z = 22$. 