**Math 311 Lecture 20**

**Gram-Schmidt process orthonormalization**

For easier reading, we again write \( u \cdot v \) instead of \( (u, v) \).

**Theorem.** If \( \{v_1, v_2, ..., v_n\} \) is an orthonormal basis, then for any \( v \),

\[
  v = c_1 v_1 + c_2 v_2 + ... + c_n v_n \quad \text{where} \quad c_i = v \cdot v_i.
\]

**Proof.** \( \{v_1, v_2, ..., v_n\} \) orthonormal implies \( v_i \cdot v_j = 1 \) (they are unit vectors) and \( v_i \cdot v_k = 0 \) if \( i \neq k \) (they are orthogonal).

\[
  v = c_1 v_1 + +c_2 v_2 + ... + c_n v_n \quad \text{for some} \quad c_i \text{'s} \quad \text{(v's are a basis)}.
\]

\[
  v \cdot v = (c_1 v_1 + +c_2 v_2 + ... + c_n v_n) v = (c_1 v_1 v_i + +c_2 v_2 v_i + ... + c_n v_n v_i) = c_1^2 (v_1 v_1) + c_2^2 (v_2 v_2) + ... + c_n^2 (v_n v_n) = c_1^2 (0) + c_2^2 (0) + ... + c_n^2 (1) + ... + c_n^2 (0) = c_i.
\]

Hence for an orthonormal basis \( S = \{v_1, v_2, ..., v_n\} \), the coordinate vector \( [v]_S \) is:

\[
  [v]_S = [c_1, c_2, ..., c_n]^T = [v \cdot v_1, v \cdot v_2, ..., v \cdot v_n]^T.
\]

If \( u \) and \( v \) are orthogonal i.e., \( (u, v) = 0 \), then are \( u \) and \( cv \)

since \( u \cdot (cv) = c (u \cdot v) = c (0) = 0 \). Multiplying by a positive scalar does not change the direction or angle.

- **Given an orthogonal basis \( \{v_1, v_2\} = \{[\frac{1}{2}, \frac{1}{2}, 0], [0, 0, -5]\} \) for a subspace \( W \), find an orthogonal basis.

  Answer: \( \{[1, 1, 0], [0, 0, -1]\} \). Note, don't factor out -1.

For any vector \( v \), \( v / ||v|| \) is the unit vector in the same direction as \( v \). We say \( v \) normalizes to \( v / ||v|| \).

- **Given an orthogonal basis \( \{v_1, v_2\} = \{[\frac{1}{2}, \frac{1}{2}, 0], [0, 0, -5]\} \) for a subspace \( W \), find an orthonormal basis.

  First simplify to \( \{[1, 1, 0], [0, 0, -1]\} \) as above.

  Answer: \( \{[1, 1, 0], [0, 0, -1]\} \).

**Lemma.** If \( v \) is a unit vector and \( w \) any other vector, then

\[
  w = u + x \quad \text{where} \quad x = \text{component of} \ w \text{ in the same direction as} \ v = (w \cdot v) v.
\]

\[
  u = \text{component of} \ w \text{ orthogonal to} \ v = w - (w \cdot v) v.
\]

\[
  \cos \theta = (w \cdot v) / ||w|| ||v|| = (w \cdot v) / ||w|| \quad \text{since} \ ||v|| = 1.
\]

Using trigonometry, \( \cos \theta = ||w|| (w \cdot v) / ||w|| \).

\[
  \therefore ||x|| = ||w|| \cos \theta = ||w|| (w \cdot v) / ||w|| = w \cdot v.
\]

Since \( x \) points in the direction of \( v \) and has length \( w \cdot v \), \( x = (w \cdot v) v \).

Subtracting \( x \) from \( w \) gives \( u \) which is \( \perp \) to \( v \).

- **Find the component of \ w = [2, 3, 4] \ which is perpendicular to the unit vector \ v = [0, 0, 1].

  \( x = (w \cdot v) v = 4 v = [0, 0, 4] \) (component in the same direction)

  \( u = w - x = [2, 3, 4] - [0, 0, 4] = [2, 3, 0] \).

  Answer: \( [2, 3, 0] \) is the component of \( w \perp \) to \( v \).

**Gram-Schmidt orthonormalization process.** Any basis \( \{w_1, w_2, ..., w_n\} \) for a subspace can be converted into an orthonormal basis \( \{u_1, u_2, ..., u_n\} \) as follows:

Let \( u_i = w_i / ||w_i|| \). Then for \( i = 1, 2, 3, ... \)

Let \( v_{i+1} = w_{i+1} - (w_{i+1} \cdot u_i) u_i - (w_{i+1} \cdot u_i) u_i - ... - (w_{i+1} \cdot u_i) u_i \)

Simplify \( v_{i+1} \) and let \( u_{i+1} = v_{i+1} / ||v_{i+1}|| \).

- **Orthonormalize the following basis for \( \mathbb{R}_3 \): \( \{w_1, w_2, w_3\} = \{[0, 0, 3], [2, 2, 2], [0, 1, 0]\} \).**

  First simplify to (scalar multiples have the same direction)

  \( \{w_1, w_2, w_3\} = \{[0, 0, 1], [1, 1, 1], [0, 1, 0]\} \)

  \( u_1 = w_1 / ||w_1|| = [0, 0, 1] / ||[0, 0, 1]|| = [0, 0, 1] / 1 = [0, 0, 1] \)

  \( v_2 = w_2 - (w_2 \cdot u_1) u_1 = [1, 1, 1] - ([1, 1, 1] \cdot [0, 0, 1]) [0, 0, 1] = [1, 1, 1] - [(1) [0, 0, 1] = [1, 1, 0] \)

  \( u_2 = v_2 / ||v_2|| = [1, 1, 0] / ||[1, 1, 0]|| = \frac{1}{\sqrt{2}} [1, 1, 0] \)

  \( v_3 = w_3 - (w_3 \cdot u_1) u_1 - (w_3 \cdot u_2) u_2 = [0, 1, 0] - ([0, 1, 0] \cdot [0, 0, 1]) [0, 0, 1] = [0, 1, 0] - ([0, 1, 0] \cdot \frac{1}{\sqrt{2}} [1, 1, 0]) = [0, 1, 0] - \frac{1}{\sqrt{2}}[1, 1, 0] = [0, 1, 0] - \frac{1}{\sqrt{2}}[1, 1, 0] = [-\frac{1}{2}, \frac{1}{2}, 0] \)

  Simplify to \( v_3 = [-1, 1, 0] \).

  \( u_3 = v_3 / ||v_3|| = [-1, 1, 0] / ||[-1, 1, 0]|| = \frac{1}{\sqrt{2}} [-1, 1, 0] \)

Hence the orthonormal basis is \( \{[0, 0, 1], \frac{1}{\sqrt{2}} [1, 1, 0], \frac{1}{\sqrt{2}} [-1, 1, 0]\} \)

Since this can be a long error-prone process, check the final basis vectors for orthogonality. Check:

\[
  [0, 0, 1] \cdot \frac{1}{\sqrt{2}} [1, 1, 0] = 0
\]

\[
  [0, 0, 1] \cdot \frac{1}{\sqrt{2}} [-1, 1, 0] = 0
\]

\[
  \frac{1}{\sqrt{2}} [1, 1, 0] \cdot \frac{1}{\sqrt{2}} [-1, 1, 0] = 0
\]

Applying the process to a dependent set will produce some zero vectors. Just delete them.

**Theorem.** In any inner product space, if \( U = \{u_1, u_2, ..., u_n\} \) is an orthonormal basis, then for any vectors \( v, w \) \( (v, w) = [v]_U \cdot [w]_U \). That is, the inner product is just the usual dot product of the coordinate vectors with respect to the basis \( U \).

**Proof.** Omitted.

---

**Hw 18 Answers**

Page 198.

2(3). \( [3, -8, -1]^T \) (a) \( [3, -8, -1]^T \) (b) \( [0, 0, 0]^T \) (c) \( [4, 4, 8]^T \)

12(2). \( \frac{1}{\sqrt{2}} \sqrt{478} \)

14(2). \( \sqrt{150} = 5 \sqrt{6} \)

16(2). \( 39 \)

18a(2). \( 3x - 2y + 4z = -16 \)

20b(2). \( x = t, y = 1 - 2t, z = t \)

22(2). \( -\frac{17}{5}, \frac{38}{5}, -6 \)