Math 311  Lecture 26

**Definition.** Let \( L:V \to W \) be an isomorphism iff \( L \) is 1-1 and onto linear transformation. \( V \) is isomorphic to \( W \), \( V \cong W \), iff \( L:V \to W \) for some isomorphism \( L \).

- Let \( L:R^3 \to R_3 \) by \( L(a_1b + b_1c + c_1d) = [a, b, c, d] \). \( L \) is 1-1, onto and linear. Hence \( L \) is an isomorphism.
- \( R_3 \) and \( R^3 \) are isomorphic via the isomorphism \( L([a, b, c]) = [a, b, c] \).

More generally, for any \( n \)-dimensional vector space \( V \) with basis \( S \), the map \( L:V \to R^n \) defined by \( L(v) = [v]_S \) is an isomorphism.

**Recall:** A linear transformation \( L \) is 1-1 iff \( \ker L = \{0\} \).
A linear transformation is uniquely determined by its values (which may be set arbitrarily) on a basis.

**Theorem.** Any two \( n \)-dimensional vector spaces are isomorphic.

**Proof.** Suppose \( V \) and \( W \) are \( n \)-dimensional vector spaces. Let \( \{v_1, v_2, ..., v_n\} \) be a basis for \( V \) and let \( \{v_1, v_2, ..., v_n\} \) be a basis for \( W \).
Let \( L:V \to W \) be the linear transformation such that \( L(v_1) = v_2, L(v_2) = v_3, ..., L(v_n) = v_n \).
Onto: \( L \) is onto since the range of \( L \) includes \( \{v_1, v_2, ..., v_n\} \), which, being a basis, spans \( W \).
1-1: Suppose \( v \in \ker L \). Then \( v \) is a linear combination \( a_1v_1 + a_2v_2 + ... + a_nv_n \) of the basis elements \( \{v_1, v_2, ..., v_n\} \).
Thus \( v \in \ker L \Rightarrow L(v) = 0 \Rightarrow L(a_1v_1 + a_2v_2 + ... + a_nv_n) = 0 \Rightarrow a_1L(v_1) + a_2L(v_2) + ... + a_nL(v_n) = 0 \Rightarrow a_1w_1 + a_2w_2 + ... + a_nw_n = 0 \Rightarrow a_i = 0 \Rightarrow v = 0 \). Thus \( \ker L = \{0\} \) and hence \( L \) is 1-1.

**Corollary.** Every \( n \)-dimensional space is isomorphic to \( R^n \).

**Theorem.** For any vector spaces \( U, V, W \), \( \cong \) is is

- **Reflexive:** \( V \cong V \),
- **Symmetric:** \( V \cong W \Rightarrow W \cong V \),
- **Transitive:** \( U \cong V \) and \( V \cong W \) implies \( U \cong W \).

**Proof.** Reflexive: Let \( I:V \to V \) by \( I(v) = v \). This the identity isomorphism.
Symmetric: If \( L:V \to W \) is an isomorphism, then, since \( L \) is 1-1, it has an inverse \( L^{-1}:W \to V \) which is also an isomorphism.
Transitive: If \( K:U \to V \) and \( L:V \to W \) are isomorphisms between \( U \) and \( V \) and between \( V \) and \( W \), then \( L \circ K:U \to W \) is an isomorphism between \( U \) and \( W \). \( \square \)

**Recall.** If \( S \) and \( T \) are bases of a vector space \( V \) and \( P_{T \leftarrow S} \) is the transition matrix from \( S \) to \( T \), then for any \( v \in V \), \( [v]_T = P_{T \leftarrow S}[v]_S \).

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**Change-of-basis Theorem.** Suppose \( L:V \to W \) is a linear transformation. Suppose \( S \) and \( S' \) are bases for \( V \) and \( T \) and \( T' \) are bases for \( W \). Suppose \( L_{S,T} \) is the matrix for \( L \) w.r.t. \( S \) and \( T \). Suppose \( L_{S',T'} \) is the matrix for \( L \) w.r.t. \( S' \) and \( T' \). Then
\[
L_{S',T'} = P_{T' \leftarrow T} \cdot L_{S,T} \cdot P_{S' \leftarrow S}.
\]

**Proof.** Let \( V_S \) be the vector space of coordinate vectors \( [v]_S \) w.r.t. the basis \( S \). Likewise for \( V_{S'}, W_T, W_{T'} \).

The following picture then makes the theorem clear.

![Diagram](image)

Suppose \( L:R_2 \to R_2 \) by \( L[x, y] = [y, x] \). Let \( U \) be the standard basis and \( T = \{(1,1), (0,1)\} \). Find \( L_{U,U}, L_{U,T}, L_{T,U}, L_{T,T} \).

First find the transition matrices \( P_{U \leftarrow T} \) and \( P_{T \leftarrow U} \).
\[
P_{U \leftarrow T} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad P_{T \leftarrow U} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},
\]
\[
L_{U,U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_{U,T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]
\[
L_{T,U} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad L_{T,T} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.
\]

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**2(5). (a)** \[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}
\]
**2(5). (b)** \[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix}
\]
**10(5). (a)** \[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1/2 & 1/2 \\ -3/2 \end{bmatrix}
\]
**14(2). (a)** \[
\begin{bmatrix} 5 & 13 \end{bmatrix} \quad \begin{bmatrix} -5 & -3 \end{bmatrix}
\]

**Hw 25 Answers**