Math 311  Lecture 40

Quadratic equations in three variables

\[ y = 2z, \ z = 2y, \ x = 2y, \ z = 2x, \ldots \] are the same up to a permutation of the variables. The vertical parabolas and horizontal parabolas \( x^2 = ay \) and \( y^2 = ax \) are the same up to a permutation of the variables. Permuting the variables may change the orientation of the graph but not the size or shape of the graph.

The dimension of a nondegenerate graph is one less than the dimension of the full space. If the variables are \( x \) and \( y \), then the full space has dimension 2 and a nondegenerate graph will have dimension 1, e.g. an ellipse. If the variables are \( x, y, \) and \( z \), the full space has dimension 3 and a nondegenerate graph will have dimension 2, e.g. a sphere.

Recall, the inertia of a form is the triple \( \text{inertia} = (\text{pos}, \text{neg}, \text{zer}) \) where \( \text{pos}, \text{neg}, \text{zer} \) are the numbers of positive, negative, and zero eigenvalues (diagonal elements of \( D \)).

Note, \( \text{pos} + \text{neg} + \text{zer} = n \), the dimension of the space.

**Theorem.** After

1. rotations to remove cross-product terms (diagonalization),
2. translations to remove nonessential terms of degree one (completing the square) and
3. a possible permutation the variables (to put them in the order listed below),

every nondegenerate quadratic equation in \( x, y, z \) can be written in one of the following six forms.

The type of surface is determined by the inertia of the associated quadratic form \( A \).

<table>
<thead>
<tr>
<th>Type of surface</th>
<th>Equation</th>
<th>Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipsoid</td>
<td>( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 )</td>
<td>(3, 0, 0)</td>
</tr>
<tr>
<td>One-sheet hyperboloid</td>
<td>( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 )</td>
<td>(2, 1, 0)</td>
</tr>
<tr>
<td>Two-sheet hyperboloid</td>
<td>( \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 )</td>
<td>(1, 2, 0)</td>
</tr>
<tr>
<td>Elliptic paraboloid</td>
<td>( \frac{x^2}{a^2} + \frac{y^2}{b^2} = z )</td>
<td>(2, 0, 1)</td>
</tr>
<tr>
<td>Hyperbolic paraboloid</td>
<td>( \frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm z )</td>
<td>(1, 1, 1)</td>
</tr>
<tr>
<td>Parabolic cylinder</td>
<td>( x^2 = ay + bz )</td>
<td>(1, 0, 2)</td>
</tr>
</tbody>
</table>

These are illustrated in the text. To get the \( x-y, x-z, \) and \( y-z \) cross sections, set \( z, y, x \) to a constant, respectively.

Other forms such as \( 0 = ax + by + cz + 1 \) (a plane), \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) (an elliptic cylinder), or \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \) (hyperbolic cone) are classified as degenerate because there are no quadratic terms (linear case), there are less than three variables or the constant is 0 (\( xy = 1 \) is a hyperbola but \( xy = 0 \) is just two lines).

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**Given that the following are nondegenerate, use inertia to classify the graph of each equation.**

- **434:4.** \( x^2 + y^2 + z^2 + 2xy = 4 \)

  Matrix for quadratic part: \( A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

  Characteristic polynomial: \( \lambda(\lambda - 1)(\lambda - 2) \)

  Eigenvalues: \( \lambda = 2, 1, 0 \).

  Inertia: (2, 0, 1).

  Classification: Elliptic paraboloid.

- **434:6.** \( 2xy + z = 0 \)

  Matrix for quadratic part: \( A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)

  Characteristic polynomial: \( \lambda(\lambda - 1)(\lambda + 1) \)

  Eigenvalues: \( \lambda = 1, -1, 0 \).

  Inertia: (1, 1, 1).

  Classification: Hyperbolic paraboloid.

- **434:8.** \( x^2 + y^2 + 2z^2 - 2xy + 4xz + 4yz = 0 \)

  Matrix for quadratic part: \( A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \)

  Characteristic polynomial: \( (\lambda - 2)(\lambda + 2)(\lambda - 4) \)

  Eigenvalues: \( \lambda = 4, 2, -2 \).

  Inertia: (2, 1, 0).

  Classification: Hyperboloid of one sheet.

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**Hw 38 Answers**

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2(1). Vertical parabola

4(1). Two parallel lines

8(1). Horizontal hyperbola

10(1). Empty

20(3). \( x^T \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} x - 1 = 0, \ \lambda_1, \lambda_2 = (-1/2)(1/2), \) hyperbola

22(3). \( x^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x - 9 = 0, \ \lambda_1, \lambda_2 = 3(-1), \) hyperbola

24(3). \( x^T \begin{bmatrix} 9 & 2 \\ 2 & 4 \end{bmatrix} x - 5 = 0, \ \lambda_1, \lambda_2 = 5(-10), \) ellipse

12(4). \( \frac{(x+z)^2}{2^2} - \frac{(y+3)^2}{2^2} = 1 \)

Horizontal hyperbola with vertices (-4, -3) and (0, -3).

14(4). \( (x - 2)^2 = -4y \)

Downward curving vertical parabola with vertex (2,0).