Math 373  Lecture 27

We want to apply \( k \) treatments to tomato plants and also to blocks of corn plants and squash plants. Let \( b \) be the number of types (blocks) of plants. Our experimental units are equal-sized plots of land, not individual plants. There are \( n = k \times b \) plots arranged in a rectangular array. There is a row or block of plots for each type of plant and a column of plots for each treatment. The plot’s response is the cash value \( x_i \) of the produce it yields. \( x_i \) depends on the plot’s treatment, its block (tomato, corn, squash), and on natural variation or error.

Let \( T_1, T_2, \ldots, T_k \) be the means of the \( k \) treatments. Warning: text uses \( T_i \) for the total.

Let \( B_1, B_2, \ldots, B_b \) be the means of the \( b \) blocks.

Let \( \bar{x} \) be the mean of all the responses \( x_i \).

Let \( = \) mean of \( \{T_1, T_2, \ldots, T_k\} \) = mean of \( \{B_1, B_2, \ldots, B_b\} \).

Let \( s_B \) be the std. dev. of \( \{B_1, B_2, \ldots, B_b\} \).

Let \( s_T \) be the std. dev. of \( \{T_1, T_2, \ldots, T_k\} \).

Let \( s_{\text{Total}} \) be the std. dev. of all \( n \) observations.

Total SS = \( \sum_{i=1}^{n} (x_i - \bar{x})^2 \) = \((n-1)s_{\text{Total}}^2\), * \( df_{\text{Total}} \) = \( n-1 \).

\( SST = \sum_{i=1}^{k} b(T_i - \bar{x})^2 \) = \( \text{(note \( T_i \) has \( b \) blocks) \( )} \)

\( b = \sum_{i=1}^{k} (T_i - \bar{x})^2 \) = \( b(k-1)s_T^2 \), * \( df_T = k-1 \).

Multiply by \( b \) since each \( T_i \) is the mean of \( b \) items.

\( MST = SST/df_T = bS_T^2 \), \( SST = MST \times df_T \).

\( SSB = \sum_{i=1}^{b} k(B_i - \bar{x})^2 \) = \( \text{(note \( B_i \) has \( k \) blocks) \( )} \)

\( k = \sum_{i=1}^{b} (B_i - \bar{x})^2 \) = \( k(b-1)s_B^2 \), * \( df_B = b-1 \),

\( MSB = SSB/df_B = ks_B^2 \), \( SSB = MSB \times df_B \),

\( SSE = \text{Total SS} - \text{SST} - \text{SSB} \), \( df_E = (b-1)(k-1) \),

\( MSE = SSE/df_E \).

\( s = \sqrt{MSE} \) = the pooled estimate of the common \( \sigma \). It has \( df_E = (b-1)(k-1) \) degrees of freedom.

How did we get SSE and \( df_E \)? The total sum of squares = the sum of squares due to treatments \( (SST) \) + the sum due to blocks \( (SSB) \) + the sum due to random error \( (SSE) \).

\( \text{Theorem. Total SS} = \text{SST} + \text{SSB} + \text{SSE}, \text{df}_{\text{Total}} = \text{df}_{\text{T}} + \text{df}_{\text{B}} + \text{df}_{\text{E}} \).

\( \therefore \text{SSE} = \text{Total SS} - \text{SST} - \text{SSB} \) and \( df_E = \text{df}_{\text{Total}} - \text{df}_{\text{T}} - \text{df}_{\text{B}} = (n-1) - (k-1) - (b-1) = b(k-1) - b + 1 = (b-1)(k-1) \).

For 4 treatments and 3 blocks, means and variances can be illustrated in a table:

<table>
<thead>
<tr>
<th>( s_{\text{Total}} ) Total SS</th>
<th>Treat 1</th>
<th>Treat 2</th>
<th>Treat 3</th>
<th>Treat 4</th>
<th>( s_B )</th>
<th>MSB</th>
<th>( df_B )</th>
<th>SSE</th>
<th>SSB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block 1</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Block 2</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Block 3</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( s_T ) MST ( df_T ) SST</td>
<td>( T_1 )</td>
<td>( T_2 )</td>
<td>( T_3 )</td>
<td>( T_4 )</td>
<td>( SSE )</td>
<td>( df_T )</td>
<td>MSE</td>
<td>( s_T )</td>
<td></td>
</tr>
</tbody>
</table>

* TI Calculators give you TotalSS, SST, directly. Don't need * lines.

To compare treatment means:
\( F = \text{MST}/\text{MSE} \) with \( df_T = k-1, df_E = (b-1)(k-1) \).

\( \text{H}_A: \) Some differ, \( \text{H}_0: \) The treatments have no effect.

\( \text{H}_A: \text{MST} > \text{MSE}, \text{H}_0: \text{MST} \leq \text{MSE} \).

\( \text{H}_A: \ F > 1, \text{H}_0: \ F = \text{MST}/\text{MSE} \leq 1 \), Null region: \( F \in [0, 1] \)

Acceptance region for \( F: F \in [0, F_a] \).

To compare block means:
\( \text{F} = \text{MSB}/\text{MSE} \) with \( df_B = b-1, df_E = (b-1)(k-1) \).

\( \text{H}_A: \) Some differ, \( \text{H}_0: \) The blocks are the same.

\( \text{H}_A: \ F > 1, \text{H}_0: \ F \leq 1 \), Null region: \( F \in [0, 1] \)

Acceptance region for \( F: F \in [0, F_a] \).

The confidence interval for the difference of treatment means \( T_i \) and \( T_j \):
\( SE = s \sqrt{\frac{1}{b} + \frac{1}{b}} = s \sqrt{\frac{2}{b}} \)

\( T_i - T_j \pm t_{\alpha/2}s \sqrt{\frac{2}{b}} = df = df_E \)

Confidence interval for a difference of two block means:
\( (b_i - b_j) \pm t_{\alpha/2}s \sqrt{\frac{2}{k}} = df = df_E \)

There are three blocks \( (1, 2, 3) \) and 2 treatments \( (A, B) \).

Fill in the values for the block and treatment means. Fill in the values for the Total SS, MST, MSB, MSE and degrees of freedom \( df_T, df_B, df_E \).

<table>
<thead>
<tr>
<th>( s_{\text{Total}} = 2.3664 ) Total SS = 28 ( k = 2 ) ( b = 3 )</th>
<th>Treat A</th>
<th>Treat B</th>
<th>( s_B = .866 )</th>
<th>MSB = 1.5</th>
<th>( df_B = 2 )</th>
<th>SSB = 3.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block 1</td>
<td>2</td>
<td>5</td>
<td>B, = 3.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Block 2</td>
<td>1</td>
<td>6</td>
<td>B, = 3.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Block 3</td>
<td>0</td>
<td>4</td>
<td>B, = 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s_T = 2.8284 ) MST = 24 ( df_T = 1 ) SST = 24</td>
<td>T, = 1</td>
<td>T, = 5</td>
<td>SSE = 1.00</td>
<td>df, = 2</td>
<td>MSE = .5</td>
<td>( s = .7071 )</td>
</tr>
</tbody>
</table>

(a) Test for an effect due to treatments.

\( \text{H}_A: \text{MST} > \text{MSE}, \text{H}_0: \text{MST} \leq \text{MSE} \), Null region: \( F \in [0, 1] \)

\( F = \text{MST}/\text{MSE} = 24/15 = 48 \quad df_T = 1 \quad df_E = 2 \quad F_a = 18.51 \quad \text{Acceptance region for } F: [0, 18.51] \)

Do some of the treatments have an effect? Yes

Why? \( F = 48 \notin [0, 18.51] \) = accept. region for no diff.

(b) Find the 95% confidence interval for the difference in means for treatments A and B.

\( (T_i - T_j) \pm t_{0.025} \sqrt{\frac{1}{b} + \frac{1}{b}} = df = df_E = 2 \)

\( (1 - 5) \pm (4.303)(.7071) \sqrt{\frac{2}{3}} = [-6.4843, -1.5157] \)